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Primitive Permutation Groups and a Characterization of the Odd Graphs

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Suppose that G is a simply transitive primitive permutation group on a finite set Ω such that for α in Ω the stabilizer G_α is 2-transitive on one of its orbits $\Gamma(\alpha)$ in Ω of length $v > 2$. Peter J. Cameron showed that G_α has another orbit related to $\Gamma(\alpha)$ of length $v(v-1)/l \geqslant 2v$. Here we show that under certain extra, fairly natural, conditions (including the assumptions that G_α is not faithful on $\Gamma(\alpha)$), G_α has a third orbit of length greater than $v(v-1)$, and obtain strong restrictions on the length of this orbit when G_α is more than 2-transitive on $\Gamma(\alpha)$. In fact if $G_\alpha^{\Gamma(\alpha)} \supseteq A_v$ and G_α is not faithful on $\Gamma(\alpha)$ we show that either the third orbit has length $v(v-1)^2$, or G is S_{2v-1} or A_{2v-1} acting as an automorphism group of the odd graph 0_v, where the set of points is identified with Ω and the set of points adjacent to α is precisely $\Gamma(\alpha)$.

Suppose that G is a simply transitive permutation group on a finite set Ω such that for α in Ω the stabilizer G_α of α acts 2-transitively on one of its orbits $\Gamma(\alpha)$ in Ω, where $|\Gamma(\alpha)| = v > 2$. Under these assumptions Peter J. Cameron (generalizing an old result of Manning [13]), showed that $\Gamma \circ \Gamma^*(\alpha) = \{\alpha; \beta \in \Omega, \Gamma(\alpha) \cap \Gamma(\beta) \neq \phi, \alpha \neq \beta\}$ is an orbit of G_α of length $v(v-1)/l \geqslant 2v$, and found stronger restrictions on l when $G_\alpha^{\Gamma(\alpha)}$ is more highly transitive (see [3, 4]). Also under these assumptions Knapp [12], and Bürker and Knapp [2] obtained information about the structure of G_α. In this paper we assume that G_α does not act faithfully on $\Gamma(\alpha)$ and, using a combination of the methods of Cameron, and Bürker and Knapp, we investigate a third orbit of G_α. First we formalise our assumptions and notation.

Hypothesis A. G is a simply transitive primitive permutation group on a finite set Ω. For $\alpha \in \Omega$, G_α has an orbit $\Gamma(\alpha)$ in Ω of length $|\Gamma(\alpha)| = v > 2$ on which it is 2-transitive and not faithful.

Notation. (a) If G is a permutation group on Ω there is a natural action of G on the cartesian product $\Omega \times \Omega$ (namely if $\alpha, \beta \in \Omega$ and $g \in G$

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then $(\alpha, \beta)^g$ is defined as $(\alpha^g, \beta^g))$. The orbits of G in $\Omega \times \Omega$ are called *orbitals* of G and a union of orbitals is called a *generalized orbital.* If Γ is a generalized orbital then $\Gamma^* = \{(\beta, \alpha); (\alpha, \beta) \in \Gamma\}$ is the generalized orbital *paired* with Γ, and if $\Gamma = \Gamma^*$, Γ is called *self-paired.* If Γ and Δ are generalized orbitals then $\Gamma \circ \Delta = \{(\alpha, \beta)$; there exists $\gamma \in \Omega$ with $(\alpha, \gamma) \in \Gamma$, $(\gamma, \beta) \in \Delta$, and $\alpha \neq \beta\}$ is also a generalized orbital. If G is transitive on Ω and $\alpha \in \Omega$ then the correspondence of an orbital Γ with the set $\Gamma(\alpha) = \{\beta; (\alpha, \beta) \in \Gamma\}$ is a one-to-one correspondence between the set of orbitals of G and the set of G_α-orbits in Ω.

(b) If Γ is an orbital of G, the *Γ-graph* is the regular directed graph whose point set is Ω and whose edges are precisely the ordered pairs in Γ. If Γ is self-paired, the Γ-graph can be regarded as an undirected graph. A connected component of the Γ-graph is a block of imprimitivity for G; G is primitive if and only if the Γ-graph is connected for each orbital Γ of G except $\{(\alpha, \alpha); \alpha \in \Omega\}$ (see [9] (1.12)).

(c) If Γ is an orbital of G and $\alpha \in \Omega$ we denote by $K(\alpha)$, $K^*(\alpha)$ the subgroups of G_α fixing $\Gamma(\alpha)$, $\Gamma^*(\alpha)$ pointwise, respectively.

(d) A permutation group G on Ω is said to be *quasiprimitive* on Ω if every nontrivial normal subgroup of G is transitive on Ω. G is *2-quasiprimitive* on Ω if G is transitive on Ω and, for α in Ω, G_α is quasiprimitive on $\Omega - \{\alpha\}$.

(e) Most other notation follows the conventions of [18]. We will sometimes refer to the alternating and symmetric groups on a set X as Alt X and Sym X, respectively. If G is a group and p is a prime $O_p(X)$ denotes the largest normal p-subgroup of G.

Very often if Hypothesis A is satisfied the following Hypothesis B is also satisfied.

Hypothesis B. For $(\alpha, \beta) \in \Gamma$, $|\Gamma \circ \Gamma^*(\alpha)| = |\Gamma^* \circ \Gamma(\alpha)| = v(v-1)$, $K(\alpha)$ is transitive on $\Gamma^*(\beta) - \{\alpha\}$, and $K^*(\beta)$ is transitive on $\Gamma(\alpha) - \{\beta\}$.

The first result gives many examples for which this is true.

THEOREM 1. *Suppose that Hypothesis* A *is satisfied.*

(a) *If both $G_\alpha^{\Gamma(\alpha)}$ and $G_\alpha^{\Gamma^*(\alpha)}$ are 2-quasiprimitive then Hypothesis* B *is true.*

(b) *If $G_\alpha^{\Gamma(\alpha)}$ is one of the known 2-transitive groups then*

(i) *Hypothesis* B *is true, or*

(ii) *$G_\alpha^{\Gamma(\alpha)}$ has a regular normal subgroup, or*

(iii) *$G_\alpha^{\Gamma(\alpha)}$ is a normal extension of $PSU(3, q)$ or $PSL(r, q)$ in their natural representations for some $r \geqslant 3$ and prime power q.*

(c) *If* $G_\alpha^{\Gamma(\alpha)}$ *is a simple group then*

(i) *Hypothesis* B *is true, or*

(ii) v *is even and if* $\beta \in \Gamma(\alpha)$ *then* $G_{\alpha\beta}^{\Gamma(\alpha)}$ *has a nontrivial noncyclic abelian normal p-subgroup, for some odd prime p, which acts semiregularly on* $\Gamma(\alpha) - \{\beta\}$,

(iii) $G_\alpha^{\Gamma(\alpha)}$ *is* $PSU(3, q)$ *or* $PSL(r, q)$ *in their natural representations for some* $r \geqslant 3$ *and prime power* q.

Remarks. After proving Theorem 1 in the first section we shall discuss the situation where Hypothesis A is true and $G_\alpha^{\Gamma(\alpha)}$ is a normal extension of $PSU(3, q)$ or $PSL(r, q)$, $r \geqslant 3$; see Lemmas 1.9 and 1.10. There was a mistake in the proof of Theorem 1 of [17]. A modified (and correct) statement of this result is proved as Lemma 1.11.

If we now assume that Hypotheses A and B are true we can show that for all but one group, $G_\alpha (\alpha \in \Omega)$ has an orbit in Ω of length greater than $v(v-1)$, and we obtain information about this orbit similar to the information which Peter J. Cameron obtained about $\Gamma \circ \Gamma^*(\alpha)$ in [4].

THEOREM 2. *If Hypotheses A and B are true then* $G_\alpha(\alpha \in \Omega)$ *is transitive on* $\Gamma_3(\alpha) = (\Gamma \circ \Gamma^* \circ \Gamma)(\alpha) - \Gamma(\alpha)$ *of degree* $v(v-1)^2/k$ *where either*

(a) G *is* S_5 *acting on the set of unordered pairs of distinct points of a set of size* 5, *and* $v = k + 1 = 3$, *or*

(b) $k < v - 1$ *and* k *divides* $v(v-1)$.

Remarks. This theorem will be proved in Section 2. In case (a), Γ is self-paired, $\Gamma_3(\alpha) = \Gamma \circ \Gamma(\alpha)$, and the Γ-graph is isomorphic to the odd graph 0_3 (that is the set of points are subsets of size 2 of a set of size 5 and two points are joined if and only if the corresponding subsets are disjoint).

If $2 < k < v - 1$ then G_α is an automorphism group of a nondegenerate balanced incomplete block design $\mathscr{D}$ with $\Gamma(\alpha)$ as the point set, $\Gamma_3(\alpha)$ as the block set, and such that a point $\beta \in \Gamma(\alpha)$ is incident with a block $\delta \in \Gamma_3(\alpha)$ if and only if $(\beta, \delta) \in \Gamma^* \circ \Gamma$. This design is discussed in Section 2. We get stronger bounds on k if $G_\alpha^{\Gamma(\alpha)}$ is more highly transitive.

THEOREM 3. *Suppose that Hypotheses* A, *and* B *are true and that* $v \geqslant 4$.

(a) *If* G_α *is* 4-*transitive on* $\Gamma(\alpha)$ *then* k *is* 1 *or* 2.

(b) *If* G_α *is* 3-*transitive on* $\Gamma(\alpha)$ *then*

(i) k *is* 1 *or* 2, *or*

(ii) $v = 4(\lambda + 1)$, $k = 2(\lambda + 1)$, *for some positive integer* λ, *or*

(iii) $v = (\lambda + 1)(\lambda^2 + 5\lambda + 5)$, $k = (\lambda + 1)(\lambda + 2)$ *for some positive integer* λ.

This theorem will be proved in Section 3. In cases (b)(ii) and (iii) G_α is an automorphism group of a symmetric 3-design which is 3-transitive on the set of points. The design is discussed in Section 3, and a discussion of the existence of such designs can be found in [11, Section 9]. Finally if $G_\alpha^{\Gamma(\alpha)}$ contains the alternating group we get the following result.

THEOREM 4. *Suppose that Hypothesis* A *is satisfied and that* $G_\alpha^{\Gamma(\alpha)} \supseteq A_v$. *Then either*

(a) $|\Gamma_3(\alpha)| = v(v-1)^2$, *or*

(b) G *is* A_{2v-1} *or* S_{2v-1}, Γ *is self-paired, and the* Γ*-graph is the odd graph* 0_v.

Recall that the odd graph 0_v has the subsets of size $v-1$ of a set of size $2v-1$ as points, and two points are joined by an edge if and only if the corresponding subsets are disjoint. Theorem 4 will be proved in Section 4.

1. A DISCUSSION OF HYPOTHESIS B AND A PROOF OF THEOREM 1

It is straightforward to prove Theorem 1(a) so we shall do that first.

Proof of Theorem 1(a). Assume that Hypothesis A is satisfied and that G_α is 2-quasiprimitive on $\Gamma(\alpha)$. By [12, Theorem 3.2] $G_\alpha^{\Gamma(\alpha)} \simeq G_\alpha^{\Gamma^*(\alpha)}$ so that $|K^*(\alpha)| = |K(\alpha)| > 1$. Let $\beta \in \Gamma(\alpha)$. Then $K^*(\beta)$ is a nontrivial normal subgroup of $G_{\alpha\beta}$ which, by [12, Theorem 5.2(3a)], acts nontrivially on $\Gamma(\alpha) - \{\beta\}$. Since $G_{\alpha\beta}$ is quasiprimitive on $\Gamma(\alpha) - \{\beta\}$, $K^*(\beta)$ is transitive on $\Gamma(\alpha) - \{\beta\}$. By [12, Theorem 5.2(3b)] $|\Gamma \circ \Gamma^*(\alpha)| = v(v-1)$. Similarly if G_α is 2-quasiprimitive on $\Gamma^*(\alpha)$ and $\beta \in \Gamma(\alpha)$, then $K(\alpha)$ is transitive on $\Gamma^*(\beta) - \{\alpha\}$. Thus Theorem 1(a) is proved.

Before discussing and proving the other parts of Theorem 1 we give a list of the known 2-transitive groups which do not have a regular normal subgroup (see [11, p. 85–86]) (see List 1.1).

From the work of Peter J. Cameron and Wolfgang Knapp we obtain immediately the following information relating to the proof of Theorem 1.

(1.2) LEMMA. *Assume that Hypothesis* A *is true and that* $G_\alpha^{\Gamma(\alpha)}$ *is either a simple group or one of the known* 2*-transitive groups with no regular normal subgroup. Then*

(a) $G_\alpha^{\Gamma^*(\alpha)} \simeq G_\alpha^{\Gamma(\alpha)}$, G_α *is* 2*-transitive on* $\Gamma^*(\alpha)$, *and* $K(\alpha) = K^*(\alpha)$.

(b) *Let* $\beta \in \Gamma(\alpha)$. *Then* $E(\alpha,\beta) = K(\alpha) \cap K(\beta)$ *is either trivial or a* p*-group where* p *is a local prime of* $G_{\alpha\beta}^{\Gamma(\alpha)}$ *and of* $G_{\alpha\beta}^{\Gamma^*(\alpha)}$ (*that is* $G_{\alpha\beta}^{\Gamma(\alpha)}$ *and* $G_{\alpha\beta}^{\Gamma^*(\alpha)}$ *have nontrivial normal* p*-subgroups*).

LIST 1.1

LIST OF KNOWN 2-TRANSITIVE PERMUTATION GROUPS G OF DEGREE v; WHERE G HAS NO REGULAR NORMAL SUBGROUP

	Group G	Degree v
1	S_v, A_v	$v \geqslant 5$
2	$PSL(2,q) \leqslant G \leqslant P\Gamma L(2,q)$	$v=q+1 \geqslant 5$, q a prime power
3	$PSL(r,q) \leqslant G \leqslant P\Gamma L(r,q)$	$v=(q^r-1)/(q-1)$, q a prime power, $r \geqslant 3$
4	$PSU(3,q) \leqslant G \leqslant P\Gamma U(3,q)$	$v=q^3+1>9$, q a prime power
5	$R(q) \leqslant G \leqslant \operatorname{Aut} R(q)$ where $R(q)$ is a group of Ree type	$v=q^3+1$, $q=3^{2m+1} \geqslant 3$
6	$Sz(q) \leqslant G \leqslant \operatorname{Aut} Sz(q)$	$v=q^2+1$, $q=2^{2m+1} \geqslant 8$
7	$Sp(2m,2)$	$v=2^{m-1}(2^m \pm 1) \geqslant 6$
8	The Mathieu groups M_v, and $\operatorname{Aut} M_{22}$	$v=11, 12, 22, 23, 24$
9	M_{11}	$v=12$
10	$PSL(2,11)$	$v=11$
11	A_7	$v=15$
12	HiS	$v=176$
13	Co_3	$v=276$

Proof. The proof follows from [12, Theorems 3.2, 3.5, and 4.11]. Immediately from Lemma 1.2(a) we deduce:

(1.3) COROLLARY. *Theorem* 1(b) *is true if* $G_\alpha^{\Gamma(\alpha)}$ *is one of the groups in lines* 1, 2, 7, 8, 9, 10, 11, 12, *or* 13 *of List* 1.1.

Proof. By Lemma 1.2, $G_\alpha^{\Gamma(\alpha)} \simeq G_\alpha^{\Gamma^*(\alpha)}$ and G_α is 2-transitive on $\Gamma^*(\alpha)$ of degree v. As all representations of degree v of the groups in lines 1, 2, 7, 8, 9, 10, 11, 12, 13 are 2-quasiprimitive, Hypothesis B is true by Theorem 1(a).

Of the remaining 2-transitive groups in List 1.1, extensions of Suzuki groups and groups of Ree type have been dealt with effectively by Bürker and Knapp.

(1.4) LEMMA. *Theorem* 1(b) *is true if* $G_\alpha^{\Gamma(\alpha)}$ *is a normal extension of a Suzuki group or a group of Ree type.*

Proof. Assume that Hypothesis A is true and that $G_\alpha^{\Gamma(\alpha)}$ is a normal extension of $S_z(q)$ for $q=2^{2m+1} \geqslant 8$ or of a group of Ree type $R(q)$ for $q=3^{2m+1} \geqslant 3$. By [12, Theorems 2.6 and 3.5], if $\beta \in \Gamma(\alpha)$, $G_\alpha \simeq G_\alpha^{\Gamma(\alpha)} \curlywedge G_{\alpha\beta}^{\Gamma^*(\beta)}$ is the direct product of $G_\alpha^{\Gamma(\alpha)}$ and $G_{\alpha\beta}^{\Gamma^*(\beta)}$ with amalgamated factor group isomorphic to a section of the outer automorphism group of $S_z(q)$, $R(q)$, respectively (for a definition of this direct product see the remark following Lemma 1.7). It follows that $K(\alpha)$ has a normal subgroup acting regularly on $\Gamma(\alpha) - \{\beta\}$, and similarly that $K^*(\beta)$ is transitive on $\Gamma^*(\beta) - \{\alpha\}$. By [12, Theorem 5.2(3)(b)], $|\Gamma \circ \Gamma^*(\alpha)| = v(v-1)$. Thus Hypothesis B is true.

Thus Theorem 1(b) follows from 1.3 and 1.4. Next we use Lemma 1.2 together with standard results on 2-transitive groups and a result of Bürker and Knapp to prove Theorem 1(c).

(1.5) LEMMA. *Assume that Hypothesis* A *is true and that* $G_\alpha^{\Gamma(\alpha)}$ *is a simple group. Then either Hypothesis* B *is true or*

(i) *v is even and if $\beta \in \Gamma(\alpha)$ then $G_{\alpha\beta}^{\Gamma(\alpha)}$ has a nontrivial noncyclic abelian normal p-subgroup, for some odd prime p, acting semiregularly on $\Gamma(\alpha) - \{\beta\}$;*

(ii) *$G_\alpha^{\Gamma(\alpha)}$ is $PSU(3, q)$ or $PSL(r, q)$ in its natural representation for some prime power q, and some $r \geqslant 3$.*

Proof. Let $\beta \in \Gamma(\alpha)$. By Lemma 1.2 either $E(\alpha,\beta) = K(\alpha) \cap K(\beta)$ is trivial or $E(\alpha,\beta)$ is a p-group and $G_{\alpha\beta}^{\Gamma(\alpha)}$ has a nontrivial normal p-subgroup, for some prime p. Suppose first that $E(\alpha,\beta)$ is nontrivial. Then $G_{\alpha\beta}^{\Gamma(\alpha)}$ must have a nontrivial abelian normal p-subgroup say N. If N is not semiregular on $\Gamma(\alpha) - \{\beta\}$ then $G_\alpha^{\Gamma(\alpha)} \simeq PSL(r, q)$ where q is a power of p and $r \geqslant 3$, by [14]. So assume that N is semiregular on $\Gamma(\alpha) - \{\beta\}$. If $|N|$ is even then by [6] $G_\alpha^{\Gamma(\alpha)}$ is $PSL(2, q)$, $S_z(q)$, or $PSU(3, q)$ where q is a power of 2. In the first two cases Hypothesis B is true by Corollary 1.3 and Lemma 1.4, respectively. So assume that p is odd. If N is cyclic then by a result of Aschbacher and O'Nan (see [11, 5B(3)]), $G_\alpha^{\Gamma(\alpha)}$ is $PSL(2, q)$, or $PSU(3, q)$ where q is a power of p, or $P\Gamma L(2, 8) = R(3)$ where $p = 3$. Again Hypothesis B is true in the first and third cases by Corollary 1.3 and Lemma 1.4. So assume that N is not cyclic. By [16, Theorem B] it follows that v is even. Thus Lemma 1.5 is true if $E(\alpha,\beta) \neq 1$.

Assume that $E(\alpha,\beta) = 1$. We wish to apply Korollar 1.2 of [2]. All the conditions are satisfied unless $G_{\alpha\beta}^{\Gamma^*(\beta)}$ has nontrivial centre. If the centre of $G_{\alpha\beta}^{\Gamma^*(\beta)}$ is nontrivial then, since $G_\alpha^{\Gamma(\alpha)} \simeq G_\alpha^{\Gamma^*(\alpha)}$, $G_{\alpha\beta}^{\Gamma(\alpha)}$ has a nontrivial abelian normal subgroup and by the arguments of the previous paragraph, Lemma 1.5 is true. Thus we may assume that all conditions of [2, Korollar 1.2] are satisfied. Hence $G_\alpha \simeq G_\alpha^{\Gamma(\alpha)} \times G_{\alpha\beta}^{\Gamma^*(\beta)}$, and it follows that $K(\alpha)$ is transitive on $\Gamma(\alpha) - \{\beta\}$ and $K^*(\beta)$ is transitive on $\Gamma^*(\beta) - \{\alpha\}$. By [12, Theorem 5.2 (3)(b)], $|\Gamma \circ \Gamma^*(\alpha)| = v(v-1)$ and so Hypothesis B is true.

The proof of Theorem 1 is completed by Lemma 1.5. However, for a satisfactory discussion of this problem we should consider the cases where $G_\alpha^{\Gamma(\alpha)}$ is a normal extension of $PSU(3, q)$ and $PSL(r, q)$, $r \geqslant 3$, more closely. If, for $(\alpha,\beta) \in \Gamma$, $E(\alpha,\beta) = K(\alpha) \cap K^*(\beta)$ is trivial then we are able to apply the result [2] Theorem 1.1 of Bürker and Knapp to show that Hypothesis B is true. By proving a small generalization of this result of Bürker and Knapp we can derive Hypothesis B if $E(\alpha, \beta)$ is only assumed to be a normal subgroup of G_α. Thus the derivability (or truth?) of Hypothesis B seems to depend more on the action of G_α on $\Gamma(\alpha) \cup \Gamma \circ \Gamma^*(\alpha)$ than on the structure of G_α. First, however, we prove the following result, which indicates the somewhat special nature of the groups $PSL(r, q)$, $r \geqslant 3$, in our situation.

(1.6) LEMMA. *Assume that Hypothesis* A *is true and let* $(\alpha, \beta) \in \Gamma$. *Then one of the following is true, where* $E(\alpha, \beta) = K(\alpha) \cap K^*(\beta)$.

(a) $E(\alpha, \beta)$ *is a normal subgroup of* G_α.

(b) $E(\alpha, \beta)$ *is not a normal subgroup of* G_α *but* $|\Gamma \circ \Gamma^*(\alpha)| = v(v-1)$,

(c) $PSL(r, q) \subseteq G_\alpha^{\Gamma(\alpha)} \subseteq P\Gamma L(r, q)$ *in its natural representation on the points of the projective space, where* $r \geqslant 3$ *and* q *is a prime power*; $|\Gamma \circ \Gamma^*(\alpha)| = v(v-1)/(q+1)$, *and if* $\delta \in \Gamma \circ \Gamma^*(\alpha)$ *then* $\Gamma(\alpha) \cap \Gamma(\delta)$ *is a line of the projective space.*

Proof. Suppose that $E = E(\alpha, \beta)$ is not a normal subgroup of G_α and that $|\Gamma \circ \Gamma^*(\alpha)| < v(v-1)$. Then for some $\beta' \in \Gamma(\alpha) - \{\beta\}$, E acts nontrivially on $\Gamma^*(\beta')$. Since $G_{\alpha\beta}$ is transitive on $\Gamma(\alpha) - \{\beta\}$, E is nontrivial on $\Gamma^*(\beta')$ for all $\beta' \in \Gamma(\alpha) - \{\beta\}$. As $|\Gamma \circ \Gamma^*(\alpha)| < v(v-1)$ there is a point $\beta' \in \Gamma(\alpha) - \{\beta\}$ such that $|\Gamma^*(\beta) \cap \Gamma^*(\beta')| > 1$. Then $K(\alpha)^{\Gamma^*(\beta')}$ is a nontrivial normal subgroup of $G_{\alpha\beta'}^{\Gamma^*(\beta')}$ containing $E^{\Gamma^*(\beta')}$. As $K(\alpha)$ fixes β and β', $\Gamma^*(\beta) \cap \Gamma^*(\beta') - \{\alpha\}$ is a nonempty union of orbits of $K(\alpha)$ which is fixed pointwise by E. Thus $K(\alpha)$ does not act faithfully on its orbits in $\Gamma^*(\beta') - \{\alpha\}$, and so by [15, Proposition 4], $G_{\beta'}^{\Gamma^*(\beta')}$ is a normal extension of $PSL(r, q)$ in its natural representation for some $r \geqslant 3$ and prime power q. As $G_\alpha^{\Gamma(\alpha)} \simeq G_\alpha^{\Gamma^*(\alpha)}$ we may assume that $G_\alpha^{\Gamma(\alpha)}$ is a normal extension of $PSL(r, q)$ in its action on the points of the projective space. By [17] (where the incorrect result, Lemma 1, is not used until Proposition 7), $|\Gamma \circ \Gamma^*(\alpha)| = v(v+1)/l$ where

(i) $l = 2$, or

(ii) $l = q$ and for $\delta \in \Gamma \circ \Gamma^*(\alpha)$, $\Gamma(\delta) \cap \Gamma(\alpha)$ is a line with a point removed, or

(iii) $l = q + 1$ and for $\delta \in \Gamma \circ \Gamma^*(\alpha)$, $\Gamma(\delta) \cap \Gamma(\alpha)$ is a line.

Suppose that $l = 2$. Let $\delta \in \Gamma \circ \Gamma^*(\alpha)$ and let $\Gamma(\delta) \cap \Gamma(\alpha) = \{\beta, \gamma\}$. Then $\Gamma^*(\beta) \cap \Gamma^*(\gamma) = \{\alpha, \delta\}$ and $G_{\alpha\{\beta,\gamma\}} = G_{\alpha\delta}$. Since $K(\alpha)$ fixes β and γ, $K(\alpha)$ fixes δ. Thus $K(\alpha)$ fixes $\Gamma \circ \Gamma^*(\alpha)$ pointwise which contradicts [12, Theorem 5.2 (3)(a)].

Suppose that $l = q$. Let $\delta \in \Gamma \circ \Gamma^*(\alpha)$ and let $\beta \in \Gamma(\alpha)$ be such that $\{\beta\} \cup (\Gamma(\delta) \cap \Gamma(\alpha))$ is a line L of the projective space. The action of G_α on $\Gamma \circ \Gamma^*(\alpha)$ is equivalent to the action of G_α on $\{(\beta, L);\ \beta \in \Gamma(\alpha),\ L$ a line,

$\beta \in L\}$. If $\gamma \in \Gamma(\alpha) \cap \Gamma(\delta)$ then $K(\alpha) \subseteq G_{\alpha\beta\gamma} \subseteq G_{\alpha\beta L} = G_{\alpha\delta}$ and it follows that $K(\alpha)$ fixes $\Gamma \circ \Gamma^*(\alpha)$ pointwise. This contradicts [12, Theorem 5.2 (3)(a)].

Thus $l = q + 1$ and Lemma 1.6 is proved. (Note that we cannot use a similar argument to eliminate this case as the action of G_α on the set of lines of the projective space is equivalent to its action of blocks of imprimitivity of length q in $\Gamma \circ \Gamma^*(\alpha)$.)

Next we derive a generalization of [2, Theorem 1.1] of Bürker and Knapp.

(1.7) LEMMA. *Let G be a transitive permutation group on a finite set Ω and let Γ be an orbital of G such that for α in Ω, $G_\alpha^{\Gamma(\alpha)}$ is quasiprimitive and $|\Gamma(\alpha)| > 1$. Let $(\alpha, \beta) \in \Gamma$ and assume that $G_{\alpha\beta}^{\Gamma^*(\beta)}$ satisfies the following hypothesis.*

(*) *For each nontrivial normal subgroup N of $X = G_{\alpha\beta}^{\Gamma^*(\beta)}$ such that $C_X(N) \neq 1$, the factor group $X/NC_X(N)$ is not isomorphic to a factor group of $G_\alpha^{\Gamma(\alpha)}$ of order less than $|G_\alpha^{\Gamma(\alpha)}|$.*

Further, let $L = \bigcap \{K^*(\beta); \beta \in \Gamma(\alpha)\}$ be the subgroup of G_α fixing $\Gamma \circ \Gamma^*(\alpha)$ pointwise and suppose that

(**) *G_α has a normal subgroup M such that $L \subset M \subseteq K(\alpha)$, $L \neq M$, and the centralizer of M/L in G_α/L is not contained in $K(\alpha)/L$.*

Then either

(a) *$L = K^*(\beta)$ for $\beta \in \Gamma(\alpha)$, or*

(b) *G_α/L is isomorphic to $G_\alpha^{\Gamma(\alpha)} \curlywedge G_\alpha/C$, the direct product of $G_\alpha^{\Gamma(\alpha)}$ and G_α/C with amalgamated factor group isomorphic to $G_\alpha/K(\alpha)C$, where C is the normal subgroup of G_α containing L such that C/L is the centralizer of M/L in G_α/L. Further* (i) *$C \cap K(\alpha) = L$, and* (ii) *for $\beta \in \Gamma(\alpha)$, $C_\beta \subseteq K^*(\beta)$ and $G_\alpha/C \simeq G_{\alpha\beta}/C_\beta$* has a factor group $G_{\alpha\beta}/K^*(\beta) = G_{\alpha\beta}^{\Gamma^*(\beta)}$.

DEFINITION. If A, B, C are groups and α, β are epimorphisms of A, B onto C, respectively, then the *direct product of A and B with amalgamated factor group C* is defined as $A \curlywedge B = \{(a, b); a \in A, b \in B, \alpha(a) = \beta(b)\}$.

Before proving Lemma 1.7 we derive a corollary of it which has the result of Bürker and Knapp as an immediate consequence.

(1.8) COROLLARY. *Suppose that G, Γ are as in Lemma 1.7 and that assumption (*) is satisfied. Suppose further, for $(\alpha, \beta) \in \Gamma$, that $E(\alpha, \beta) = K(\alpha) \cap K^*(\beta)$ is a normal subgroup of G_α. Then either $E(\alpha, \beta) = K(\alpha)$, or $E(\alpha, \beta) = K^*(\beta)$, or $G_\alpha/E(\alpha, \beta)$ is isomorphic to $G_\alpha^{\Gamma(\alpha)} \curlywedge G_{\alpha\beta}^{\Gamma^*(\beta)}$, the direct product of $G_\alpha^{\Gamma(\alpha)}$ and $G_{\alpha\beta}^{\Gamma^*(\beta)}$ with amalgamated factor group of order less than $|G_{\alpha\beta}^{\Gamma^*(\beta)}|$.*

Proof of Corollary 1.8. As $E(\alpha,\beta)$ is normal in G_α, $E(\alpha,\beta)=L$ and it follows that $K(\alpha)/L$ is centralized by $K^*(\beta)/L$. Thus if $L\neq K(\alpha)$ and $L\neq K^*(\beta)$ then condition (**) of Lemma 1.7 is satisfied with $M=K(\alpha)$. Thus Lemma 1.7(b) is true, and in this case $C_\beta=K^*(\beta)$ so that $G_\alpha/C\simeq G_{\alpha\beta}/C_\beta=G_{\alpha\beta}/K^*(\beta)=G_{\alpha\beta}^{\Gamma^*(\beta)}$. Finally the amalgamated factor group $G_\alpha/K(\alpha)C$ has order less than $|G_\alpha/C|$ since $C\nsubseteq K(\alpha)$. Thus Corollary 1.8 is proved.

Proof of Lemma 1.7. Assume that $L\neq K^*(\beta)$ for $\beta\in\Gamma(\alpha)$. Let C be the normal subgroup of G_α containing L such that C/L is the centralizer in G_α/L of M/L, where M is a subgroup satisfying (**). By (**), $C\nsubseteq K(\alpha)$. Let $H=MC\cap G_{\alpha\beta}=MC_\beta$. Since $G_\alpha^{\Gamma(\alpha)}$ is quasiprimitive and $C^{\Gamma(\alpha)}=CK(\alpha)/K(\alpha)$ is a nontrivial normal subgroup, C is transitive on $\Gamma(\alpha)$. Hence $G_\alpha=G_{\alpha\beta}C$, where $\beta\in\Gamma(\alpha)$, and so $G_\alpha/MC=G_{\alpha\beta}C/MC=G_{\alpha\beta}MC/MC\simeq G_{\alpha\beta}/MC_\beta=G_{\alpha\beta}/H$. Thus $G_{\alpha\beta}/H$ is isomorphic to a factor group of $G_\alpha^{\Gamma(\alpha)}$ of order less than $|G_\alpha^{\Gamma(\alpha)}|$.

Now consider $X=G_{\alpha\beta}^{\Gamma^*(\beta)}$ with its normal subgroup $M^{\Gamma^*(\beta)}$. If $M^{\Gamma^*(\beta)}=1$ then $M\subseteq K^*(\beta)$ and as G_α is transitive on $\Gamma(\alpha)$ it follows that $M\subseteq\bigcap\{K^*(\beta);\beta\in\Gamma(\alpha)\}=L$ contrary to assumption (**). Thus $M^{\Gamma^*(\beta)}\neq 1$. Let Y be the normal subgroup of $G_{\alpha\beta}$ containing $K^*(\beta)$ such that $Y^{\Gamma^*(\beta)}=C_X(M^{\Gamma^*(\beta)})$. We claim that $C_\beta\subseteq Y$: Let $g\in C_\beta$ and $x\in M$ and consider $g^{-1}xgK^*(\beta)$; as $g\in C_\beta$, $[x,g]=x^{-1}g^{-1}xg\in L\subseteq K^*(\beta)$ so that $g^{-1}xgK^*(\beta)=xK^*(\beta)$, that is $g\in Y$. Thus $C_\beta\subseteq Y$. Now consider $G_{\alpha\beta}^{\Gamma^*(\beta)}/M^{\Gamma^*(\beta)}Y^{\Gamma^*(\beta)}\simeq G_{\alpha\beta}/MY$. Since $MY\supseteq MC_\beta=H$, $G_{\alpha\beta}/MY$ is a factor group of $G_{\alpha\beta}/H$ which is isomorphic to a factor group of $G_\alpha^{\Gamma(\alpha)}$ of order less than $|G_\alpha^{\Gamma(\alpha)}|$. It follows from assumption (*) that $Y^{\Gamma^*(\beta)}=1$, that is, $Y=K^*(\beta)$, and hence $K^*(\beta)\supseteq C_\beta$.

Thus $G_\alpha/C=G_{\alpha\beta}C/C\simeq G_{\alpha\beta}/C_\beta$ has a factor group $G_{\alpha\beta}/K^*(\beta)=G_{\alpha\beta}^{\Gamma^*(\beta)}$. Also $C\cap K(\alpha)=\bigcap\{C_\beta\cap K(\alpha);\ \beta\in\Gamma(\alpha)\}\subseteq\bigcap\{K^*(\beta)\cap K(\alpha);\ \beta\in\Gamma(\alpha)\}=L\subseteq C\cap K(\alpha)$ so that $C\cap K(\alpha)=L$. It is then straightforward to show that G_α/L is isomorphic to $G_\alpha/K(\alpha)\curlywedge G_\alpha/C$ with amalgamated factor group $G_\alpha/K(\alpha)C$. Thus Lemma 1.7 is proved.

Finally we use these results to obtain information about the structure of G_α in the two exceptional cases of Theorem 1 (that is parts (b) (iii) and (c)(iii)).

(1.9) LEMMA. *Assume that Hypothesis* A *is true and Hypothesis* B *is not true, and that* $PSU(3,q)\subseteq G_\alpha^{\Gamma(\alpha)}\subseteq \mathrm{Aut}\,PSU(3,q)$ *in its natural representation, where q is a power of a prime p. Let* $(\alpha,\beta)\in\Gamma$. *Then*

(a) $|\Gamma\circ\Gamma^*(\alpha)|=v(v-1)$,

(b) $K(\alpha)$ *is a p-group*; $K(\alpha)=O_p(G_\alpha)$,

(c) $E(\alpha,\beta)=K(\alpha)\cap K^*(\beta)$ *is not normal in* G_α; *the action of* G_α *on*

$\Gamma(\alpha)$ is equivalent to the action of G_α on the set $\{E(\alpha,\beta); \beta \in \Gamma(\alpha)\}$ by conjugation.

(d) *G_α, $G_{\alpha\beta}$, and G_α/L are strongly p-constrained where $L = \bigcap \{K^*(\beta); \beta \in \Gamma(\alpha)\}$.*

Remarks. We have more technical information about G_α, namely, the negation of condition (**) of Lemma 1.7.

A group G is said to be *strongly p-constrained* for some prime p, if $C_G(O_p(G)) \subseteq O_p(G)$.

Proof. It is easy to check that condition (*) of Lemma 1.7 is satisfied. Suppose that condition (**) of Lemma 1.7 is also satisfied. Then $G_\alpha/L \simeq G_\alpha^{\Gamma(\alpha)} \curlywedge G_\alpha/C$, with amalgamated factor group $G_\alpha/K(\alpha)C$, is isomorphic to a subgroup of the outer automorphism group of $PSU(3,q)$. It follows that $K(\alpha)$ is transitive on $\Gamma^*(\beta) - \{\alpha\}$ and similarly that $K^*(\beta)$ is transitive on $\Gamma(\alpha) - \{\beta\}$. By [12, Theorem 5.2 (3)(b)], $|\Gamma \circ \Gamma^*(\alpha)| = v(v-1)$ so that Hypothesis B is true, which contradicts our assumptions. Thus condition (**) of Lemma 1.7 does not hold. In particular $E(\alpha,\beta)$ is not normal in G_α, so that $E(\alpha,\beta)$ is nontrivial, and $|\Gamma \circ \Gamma^*(\alpha)| = v(v-1)$ by Lemma 1.6. By [12, 3.5 and 4.12], $E(\alpha,\beta)$ is a p-group and G_α and $G_{\alpha\beta}$ are strongly p-constrained. As $G_{\alpha\beta}$ is transitive on $\Gamma(\alpha) - \{\beta\}$ the subgroups $E(\alpha,\beta')$, $\beta' \in \Gamma(\alpha)$ are all distinct and it is easy to check that the actions of G_α on $\Gamma(\alpha)$ and on $\{E(\alpha,\beta); \beta \in \Gamma(\alpha)\}$ by conjugation are equivalent.

If $K(\alpha)$ is transitive on $\Gamma^*(\beta) - \{\alpha\}$ then $K^*(\beta)$ is transitive on $\Gamma(\alpha) - \{\beta\}$ and Hypothesis B is true. Thus $K(\alpha)$ is intransitive on $\Gamma^*(\beta) - \{\alpha\}$ and it is fairly easy to check that $|K(\alpha)^{\Gamma^*(\beta)}| = q$. Hence $K(\alpha)$ is a p-group and indeed $K(\alpha) = O_p(G_\alpha)$. As (**) of Lemma 1.7 is not true $C_{G\alpha/L}(K(\alpha)/L) \subseteq K(\alpha)/L$, that is, G_α/L is strongly p-constrained. Thus Lemma 1.9 is proved.

(1.10) LEMMA. *Assume that Hypothesis* A *is true and Hypothesis* B *is not true and that $PSL(r,q) \subseteq G_\alpha^{\Gamma(\alpha)} \subseteq P\Gamma L(r,q)$ in its representation on the points of the projective space, where $r \geqslant 3$ and q is a power of a prime p. Let $(\alpha,\beta) \in \Gamma$. Then*

(a) *either $|\Gamma \circ \Gamma^*(\alpha)| = v(v-1)$, or $|\Gamma \circ \Gamma^*(\alpha)| = v(v-1)/(q+1)$ and if $\delta \in \Gamma \circ \Gamma^*(\alpha)$ then $\Gamma(\alpha) \cap \Gamma(\delta)$ is a line of the projective space;*

(b) *$E(\alpha,\beta) = K(\alpha) \cap K^*(\beta)$ is a nontrivial p-group, not normal in G_α, and the action of G_α by conjugation on the set $\{E(\alpha,\beta): \beta \in \Gamma(\alpha)\}$ is equivalent to the action of G_α on $\Gamma(\alpha)$;*

(c) *G_α, $G_{\alpha\beta}$ and G_α/L are strongly p-constrained, where $L = \bigcap \{K^*(\beta); \beta \in \Gamma(\alpha)\}$.*

Proof. It is shown in [17, Lemma 1] that condition (*) of Lemma 1.7 is true. Assume that condition (**) of Lemma 1.7 is also true. Then as in the proof of Lemma 1.9, we find that Hypothesis B is true contrary to our assumptions. Thus condition (**) is not true. In particular part (a) of Lemma 1.10 is true by Lemma 1.6, $E(\alpha, \beta)$ is a nontrivial p-group, and G_α and $G_{\alpha\beta}$ are strongly p-constrained (see [12, 3.5 and 4.12]). Also as $G_{\alpha\beta}$ is transitive on $\Gamma(\alpha) - \{\beta\}$ the subgroups $E(\alpha, \beta')$, $\beta' \in \Gamma(\alpha)$, are all distinct and part (b) follows. Finally as (**) of Lemma 1.7 is not true the centralizer of $O_p(K(\alpha)/L)$ in G_α/L is contained in $K(\alpha)/L$. Also the centralizer of $O_p(K(\alpha)^{\Gamma^*(\beta)}) = O_p(K(\alpha))^{\Gamma^*(\beta)}$ in $G_{\alpha\beta}^{\Gamma^*(\beta)}$ is contained in $O_p(K(\alpha))^{\Gamma^*(\beta)}$. It is then straightforward to show that the centralizer of $O_p(K(\alpha)/L)$ in G_α/L is contained on $O_p(K(\alpha)/L)$, that is, that G_α/L is strongly p-constrained.

To conclude this section it is perhaps appropriate to state a corrected version of [17, Theorem 1]. (Theorem 2 of [17] is clearly true.)

(1.11) LEMMA. *Let G be a simply transitive permutation group on Ω. Suppose that G_α, $\alpha \in \Omega$, has an orbit $\Gamma(\alpha)$ such that $PSL(r, q) \subseteq G_\alpha^{\Gamma(\alpha)} \subseteq P\Gamma L(r, q)$ in its natural representation on the $v = |\Gamma(\alpha)| = (q^r - 1)/(q - 1)$ points of the projective space, where $r \geqslant 3$ and q is a power of a prime p. Then $\Gamma \circ \Gamma^*(\alpha)$ is an orbit of G_α of length $v(v-1)/k$ where k satisfies one of the following:*

(a) $k = 1$,

(b) $k = 2$ *and G_α is faithful on $\Gamma(\alpha)$;*

(c) $k = q$ *is 3 or 4 and if $\delta \in \Gamma \circ \Gamma^*(\alpha)$ then $\Gamma(\alpha) \cap \Gamma(\delta)$ is a line of the projective space with one point removed. Again G_α is faithful on $\Gamma(\alpha)$.*

(d) $k = q + 1$ *and if $\delta \in \Gamma \circ \Gamma^*(\alpha)$ then $\Gamma(\alpha) \cap \Gamma(\delta)$ is a line of the projective space. If G_α is faithful on $\Gamma(\alpha)$ then k is 3, 4 or 6. If G_α is not faithful on $\Gamma(\alpha)$ then the conclusions of Lemma* 1.10(b) *and* (c) *are true.*

2. PROOF OF THEOREM 2, AND A DISCUSSION OF THE DESIGN $\mathscr{D}$

Assume that both Hypotheses A and B are true.

(2.1) LEMMA. *G is transitive on the nonempty set of 4-tuples $\bar{\Gamma}_3 = \{(\alpha, \beta, \gamma, \delta); (\alpha, \beta), (\gamma, \beta), (\gamma, \delta) \in \Gamma, \alpha \neq \gamma, (\alpha, \delta) \notin \Gamma\}$.*

Proof. First we show that $\bar{\Gamma}_3$ is not empty. For if $\bar{\Gamma}_3$ is empty then, if $\gamma \in \Gamma \circ \Gamma^*(\alpha)$, $\Gamma(\gamma) = \Gamma(\alpha)$, and so $\Gamma(\alpha)$ is fixed setwise by $\langle G_\alpha, G_\gamma \rangle = G$ which is not true by [18, 8.7]. Thus $\bar{\Gamma}_3 \neq \phi$. Let $(\alpha_i, \beta_i, \gamma_i, \delta_i) \in \bar{\Gamma}_3$ for $i = 1, 2$. As G is transitive there is an element $g \in G$ such that $\alpha_1^g = \alpha_2 = \alpha$. As G_α is transitive on $\Gamma(\alpha)$ there is an element $h \in G_\alpha$ such that $\beta_1^{gh} = \beta_2$. As $K(\alpha)$ (which fixes α and β_2) is transitive on $\Gamma^*(\beta_2) - \{\alpha\}$, there is an element $k \in K(\alpha)$ such that $\gamma_1^{ghk} = \gamma_2$. Finally $K^*(\beta_2)$ (which fixes α, β_2, and γ_2) is transitive on $\Gamma(\gamma_2) - \{\beta_2\}$ and so there is an element $m \in K^*(\beta_2)$ such that

$\delta_1^{ghkm}=\delta_2$. Then $(\alpha_1,\beta_1,\gamma_1,\delta_1)^{ghkm}=(\alpha_2,\beta_2,\gamma_2,\delta_2)$ and the lemma is proved.

It follows that G_α is transitive on $\Gamma_3(\alpha)=\Gamma\circ\Gamma^*\circ\Gamma(\alpha)-\Gamma(\alpha)=\{\delta;$ for some β, γ, $(\alpha,\beta,\gamma,\delta)\in\bar{\Gamma}_3\}$. Set $\Gamma_2=\Gamma\circ\Gamma^*$, and let $\delta\in\Gamma_3(\alpha)$. Then as G_α is transitive on $\Gamma_3(\alpha)$, $k=|\Gamma^*(\delta)\cap\Gamma_2(\alpha)|$ is independent of the point δ. Counting Γ-edges from $\Gamma_2(\alpha)$ to $\Gamma_3(\alpha)$ we have $|\Gamma_3(\alpha)|=v(v-1)^2/k$.

(2.2) LEMMA. *If $\Gamma_3(\alpha)=\Gamma_2(\alpha)$ then $v=k+1=3$, Γ is self-paired, and G is S_5 acting as a rank 3 group on the set of unordered pairs of distinct points of a set of size* 5.

Proof. Let $\Gamma_3(\alpha)=\Gamma_2(\alpha)$. Then $k=v-1$ and $\Gamma_3(\alpha)$ is self-paired. Suppose that $(\alpha,\beta,\gamma,\delta)\in\bar{\Gamma}_3$. Then $\delta\in\Gamma_3(\alpha)$. Since $\Gamma_3(\alpha)=\Gamma_2(\alpha)$ there is an $\eta\in\Gamma(\alpha)\cap\Gamma(\delta)$. Also since $\Gamma_3(\alpha)=\Gamma_3^*(\alpha)$ there are points β', γ' such that (α,β'), (γ',β'), (γ',δ) all lie in Γ^*. Then $(\alpha,\eta,\delta,\gamma')\in\bar{\Gamma}_3$ so that $\gamma'\in\Gamma_3(\alpha)\cap(\Gamma^*\circ\Gamma(\alpha))$, and as $\Gamma_3(\alpha)$ and $\Gamma^*\circ\Gamma(\alpha)$ are orbits of G_α it follows that $\Gamma\circ\Gamma^*(\alpha)=\Gamma_3(\alpha)=\Gamma^*\circ\Gamma(\alpha)$. Then it is easy to check that $\{\alpha\}\cup\Gamma^*(\alpha)\cup\Gamma_3(\alpha)$ is a connected component of the $\Gamma\circ\Gamma^*$-graph. By [9, (1.12)], $\Omega=\{\alpha\}\cup\Gamma^*(\alpha)\cup\Gamma_3(\alpha)$ and it follows that Γ is self-paired and G has rank 3 with subdegrees 1, v, $v(v-1)$. By [7, Theorem 2, 8, and 1] and since G_α is not faithful on $\Gamma(\alpha)$ we conclude that $v=k+1=3$ and G is S_5 on unordered pairs.

Thus we may assume that $\Gamma_3(\alpha)\neq\Gamma_2(\alpha)$, and similarly that $\Gamma_3(\alpha)\neq\Gamma^*\circ\Gamma(\alpha)$. Now for $\delta\in\Gamma_3(\alpha)$, $k=|\Gamma^*(\delta)\cap\Gamma_2(\alpha)|\leqslant|\Gamma^*(\delta)|=v$. If $k=v$ then $\{\alpha\}\cup\Gamma_2(\alpha)$ is a connected component of the $\Gamma\circ\Gamma^*$-graph which contradicts [9, (1.12)]. Thus $k<v$.

Discussion of the design. If $k\geqslant 2$ then G_α is a group of automorphisms of a (possibly degenerate) balanced incomplete block design $\mathscr{D}$, the points of $\mathscr{D}$ being the points β of $\Gamma(\alpha)$, the blocks of $\mathscr{D}$ being the points δ of $\Gamma_3(\alpha)$ such that β is incident with δ if and only if $\Gamma^*(\beta)\cap\Gamma^*(\delta)$ is nonempty. (It is clearly a design since G_α is 2-transitive on $\Gamma(\alpha)$ and transitive on $\Gamma_3(\alpha)$).

Assume now that $k\geqslant 2$ and let $\delta\in\Gamma_3(\alpha)$. We shall set

$$(\delta)=\Gamma(\alpha)\cap\Gamma^*\circ\Gamma(\delta)$$

the set of points of $\mathscr{D}$ incident with δ, and

$$[\delta]=\Gamma^*(\delta)\cap\Gamma_2(\alpha).$$

(2.3) LEMMA. (a) *$G_{\alpha\delta}$ is transitive on (δ) and $[\delta]$ and the actions are equivalent*; $|(\delta)|=|[\delta]|=k$.

(b) *A point of $\mathscr{D}$ is incident with $(v-1)^2$ blocks of $\mathscr{D}$, and two distinct points of $\mathscr{D}$ are incident with $\lambda=(v-1)(k-1)$ common blocks.*

(c) *If* δ, δ' *are distinct points of* $\Gamma_3(\alpha)$ *then* $|[\delta] \cap [\delta']| \leqslant 1$.

(d) k *divides* $v(v-1)$.

(e) *There is an equivalence relation* $\sim$ *on* $\Gamma_3(\alpha)$ *such that for* δ, $\delta' \in \Gamma_3(\alpha)$, $\delta \sim \delta'$ *if and only if* $(\delta) = (\delta')$. *An equivalence class has size* $y(v-1)$ *where* y *divides* $(v-1, k-1)$. *The number of distinct* (δ) *for* $\delta \in \Gamma_3(\alpha)$ *is* $v(v-1)/yk$.

Proof. (a) That $G_{\alpha\delta}$ is transitive on (δ) and $[\delta]$ follows from Lemma 2.1, and $|[\delta]| = k$ by definition of k. If $\beta \in (\delta)$ then there is a point γ in $[\delta] \cap \Gamma^*(\beta)$; there is only one such point since there is only one $\Gamma^* \circ \Gamma$-edge from β to δ. Thus each $\beta \in (\delta)$ determines a unique $\gamma \in [\delta] \cap \Gamma^*(\beta)$. Further, distinct points β in (δ) determine distinct points γ in $[\delta]$ since there is only one $\Gamma \circ \Gamma^*$-edge from α to γ. Hence $|(\delta)| = k$ and the actions of $G_{\alpha\delta}$ on (δ) and $[\delta]$ are equivalent.

(b) This follows from the equations on the parameters of the design.

(c) Distinct points of $[\delta] \cap [\delta']$ determine distinct $\Gamma^* \circ \Gamma$-edges from δ to δ', and as there is at most one such edge, $|[\delta] \cap [\delta']| \leqslant 1$.

(d) Let $\delta \in \Gamma_3(\alpha)$. Then as $G_{\alpha\delta}$ acts equivalently on $[\delta]$ and (δ), and as $K(\alpha) \cap G_{\alpha\delta}$ fixes (δ) pointwise, $K(\alpha) \cap G_{\alpha\delta}$ fixes $[\delta]$ pointwise. Let $\gamma \in [\delta]$. Then $K(\alpha) \cap G_{\alpha\delta} \subseteq K(\alpha) \cap G_{\alpha\gamma}$. Also by Hypothesis B, if $\Gamma(\alpha) \cap \Gamma(\gamma) = \{\beta\}$, $K(\alpha)$ is transitive on $\Gamma^*(\beta) - \{\alpha\}$ so that $|K(\alpha) : K(\alpha) \cap G_{\alpha\gamma}| = v - 1$. Thus $v - 1$ divides $|K(\alpha) : K(\alpha) \cap G_{\alpha\delta}| = |K(\alpha) G_{\alpha\delta} : G_{\alpha\delta}|$ which divides $|G_\alpha : G_{\alpha\delta}| = v(v-1)^2/k$. Hence k divides $v(v-1)$.

(e) It is easy to check that the relation $\sim$ on $\Gamma_3(\alpha)$ is an equivalence relation. If $\beta \in \Gamma(\alpha)$ then $\Gamma^* \circ \Gamma(\beta) \cap \Gamma_3(\alpha)$ is a union of equivalence classes, and is a set of size $(v-1)^2$ by part (b). If β_1, β_2 are distinct points of $\Gamma(\alpha)$ then $(\Gamma^* \circ \Gamma(\beta_1)) \cap (\Gamma^* \circ \Gamma(\beta_2)) \cap \Gamma_3(\alpha)$ is a union of equivalence classes, and is a set of size $(v-1)(k-1)$ by part (b). Finally if $\delta \in \Gamma_3(\alpha)$ then $G_{\alpha\delta} K(\alpha)$ is contained in the setwise stabilizer X of (δ). Hence the size of an equivalence class, namely, $|X : G_{\alpha\delta}|$, is divisible by $|G_{\alpha\delta} K(\alpha) : G_{\alpha\delta}|$ which by part (d) is divisible by $v - 1$.

To complete the proof of Theorem 2 it is sufficient to show that $k < v - 1$. So suppose that $k = v - 1$.

Let $\delta \in \Gamma_3(\alpha)$. Then $\Gamma^*(\delta) = [\delta] \cup \{\eta\}$ for some point $\eta \notin \Gamma_2(\alpha)$, and $G_{\alpha\delta} \subseteq G_{\alpha\eta}$. We claim that $\eta \notin \{\alpha\} \cup \Gamma(\alpha) \cup \Gamma_2(\alpha) \cup \Gamma_3(\alpha)$. If $\gamma \in [\delta]$ then there is only one $\Gamma \circ \Gamma^*$-edge from α to γ and so $\eta \neq \alpha$; also $\Gamma_2(\gamma) \neq \Gamma_3(\gamma)$ and so $\eta \notin \Gamma(\alpha)$. If $\eta \in \Gamma_3(\alpha)$ then as $v \geqslant 3$, $(\delta) \cap (\eta)$ contains a point β say, and we have $\eta \in (\Gamma^* \circ \Gamma(\beta)) \cap \Gamma_3^*(\beta)$. It follows that $\Gamma^* \circ \Gamma(\beta) = \Gamma_3^*(\beta) = \Gamma_3(\beta)$ which contradicts our assumption. Thus η lies in an orbit of G_α different from $\{\alpha\}$, $\Gamma(\alpha)$, $\Gamma_2(\alpha)$, and $\Gamma_3(\alpha)$. Since each δ in $\Gamma_3(\alpha)$ determines a unique point η, the set of points η forms a G_α-orbit, namely,

$\Gamma_4(\alpha)=\Gamma_3\circ\Gamma^*(\alpha)-\Gamma_2(\alpha)$; counting pairs $(\delta,\eta)\in\Gamma^*$ with $\delta\in\Gamma_3(\alpha)$, $\eta\in\Gamma_4(\alpha)$, we have $|\Gamma_4(\alpha)|=v(v-1)/\ell$ where $\ell=|\Gamma_3(\alpha)\cap\Gamma(\eta)|\leqslant v$.

Choose $\eta\in\Gamma_4(\alpha)$ and $\beta\in\Gamma_3(\eta)\cap\Gamma(\alpha)$. Then there are exactly $k=v-1$ 4-tuples $(\eta,\delta,\gamma,\beta)\in\bar{\Gamma}_3$ for some points δ, γ. For such a 4-tuple it is straightforward to show that $\gamma\in\Gamma_2(\alpha)$ and $\delta\in\Gamma_3(\alpha)$. There are at most ℓ choices of δ since $\delta\in\Gamma_3(\alpha)\cap\Gamma(\eta)$, and for each δ there is a unique point γ since $(\alpha,\beta)\in\Gamma_2$. Hence $\ell\geqslant v-1$. If $\ell=v$ then $\Gamma(\eta)\subseteq\Gamma_3(\alpha)$ and $\{\alpha\}\cup\Gamma_2(\alpha)\cup\Gamma_4(\alpha)$ is a connected component of the Γ_2-graph, which contradicts [9, (1.12)]. Thus $\ell=v-1$ so that $|\Gamma_4(\alpha)|=v$. If $\delta\in\Gamma_3(\alpha)\cap\Gamma(\eta)$ then $G_{\alpha\eta}\supseteq G_{\alpha\delta}$ and $|G_{\alpha\eta}:G_{\alpha\delta}|=v-1$. Since $G_{\alpha\delta}$ is transitive on (δ) and fixes $\{\beta\}=\Gamma(\alpha)-(\delta)$, $G_{\alpha\eta}\supseteq G_{\alpha\eta\beta}\supseteq G_{\alpha\delta}$ and it follows that $G_{\alpha\eta}=G_{\alpha\eta\beta}=G_{\alpha\beta}$. Thus the actions of G_α on $\Gamma(\alpha)$ and $\Gamma_4(\alpha)$ are equivalent.

Consider $\Gamma(\eta)$ which equals $(\Gamma(\eta)\cap\Gamma_3(\alpha))\cup\{\zeta\}$ for some point ζ. Clearly $G_{\alpha\eta}$ fixes ζ and the only fixed points of $G_{\alpha\eta}$ in $\{\alpha\}\cup\Gamma(\alpha)\cup\Gamma_2(\alpha)\cup\Gamma_3(\alpha)\cup\Gamma_4(\alpha)$ are α, β and η. If $\zeta=\alpha$ then $\beta\in\Gamma_2(\eta)\cap\Gamma_3(\eta)$, and if $\zeta=\beta$ then $\eta\in\Gamma_2(\alpha)$, both of which are impossible. Thus ζ lies in a G_α-orbit distinct from $\{\alpha\}$, $\Gamma(\alpha)$, $\Gamma_2(\alpha)$, $\Gamma_3(\alpha)$, and $\Gamma_4(\alpha)$, namely, $\Gamma_5(\alpha)=\Gamma_4\circ\Gamma(\alpha)-\Gamma_3(\alpha)$. (As each η determines a unique ζ, G_α is transitive on $\Gamma_5(\alpha)$.) Counting pairs $(\eta,\zeta)\in\Gamma$ with $\eta\in\Gamma_4(\alpha)$, $\zeta\in\Gamma_5(\alpha)$ we find that $|\Gamma_5(\alpha)|=v/m$ where $m=|\Gamma^*(\zeta)\cap\Gamma_4(\alpha)|$. Now $G_{\alpha\eta}\subseteq G_{\alpha\zeta}\subseteq G_\alpha$ and $G_{\alpha\eta}$ is a maximal subgroup of G_α, and G_α fixes only the point α of Ω, we have $G_{\alpha\eta}=G_{\alpha\zeta}$. Hence $|\Gamma_5(\alpha)|=v$ and the actions of G_α on $\Gamma(\alpha)$, $\Gamma_4(\alpha)$ and $\Gamma_5(\alpha)$ are all equivalent.

Choose $\zeta\in\Gamma_5(\alpha)$ and $\gamma\in\Gamma_3^*(\zeta)\cap\Gamma_2(\alpha)$. Then there are exactly $k=v-1$ 4-tuples $(\gamma,\delta,\eta,\zeta)\in\bar{\Gamma}_3$ for some points δ, η. It is straightforward to show that $\delta\in\Gamma_3(\alpha)$ and $\eta\in\Gamma_4(\alpha)$. However, there is only one choice of $\eta\in\Gamma_4(\alpha)\cap\Gamma^*(\zeta)$, and given η there is only one choice of $\delta\in\Gamma(\alpha)\cap\Gamma(\eta)$. This is a contradiction. Thus $k<v-1$ and the proof of Theorem 2 is complete.

3. Proof of Theorem 3

Suppose that Hypotheses A and B are true, that G_α is 3-transitive on $\Gamma(\alpha)$, of degree $v\geqslant 4$, and that $k>2$ where $|\Gamma_3(\alpha)|=v(v-1)^2/k$. We set $\Gamma_2(\alpha)=\Gamma\circ\Gamma^*(\alpha)$. Let $\delta\in\Gamma_2(\alpha)$ and let $\{\beta\}=\Gamma(\alpha)\cap\Gamma(\gamma)$. Then $|G_\alpha^{\Gamma(\alpha)}:G_{\alpha\gamma}^{\Gamma(\alpha)}| = |G_\alpha:G_{\alpha\gamma}K(\alpha)| = |G_\alpha:G_{\alpha\gamma}|/|G_{\alpha\gamma}K(\alpha):G_{\alpha\gamma}|=v(v-1)/|K(\alpha):K(\alpha)\cap G_{\alpha\gamma}|=v$ (since $K(\alpha)$ is transitive on $\Gamma^*(\beta)-\{\alpha\}$). As $G_{\alpha\gamma}^{\Gamma(\alpha)}\subseteq G_{\alpha\beta}^{\Gamma(\alpha)}$ we have $G_{\alpha\gamma}^{\Gamma(\alpha)}=G_{\alpha\beta}^{\Gamma(\alpha)}$. Let $\delta\in\Gamma_3(\alpha)\cap\Gamma(\gamma)$. Then as the actions of $G_{\alpha\delta}$ on (δ) and $[\delta]$ are equivalent $G_{\alpha\delta\beta}=G_{\alpha\delta\gamma}$. Thus $z=|G_{\alpha\beta}^{\Gamma(\alpha)}:G_{\alpha\gamma\delta}^{\Gamma(\alpha)}|=|G_{\alpha\gamma}^{\Gamma(\alpha)}:G_{\alpha\gamma\delta}^{\Gamma(\alpha)}|$ divides $|G_{\alpha\gamma}:G_{\alpha\gamma\delta}|=v-1$. Now $G_{\alpha\beta}$ is 2-transitive on $\Gamma(\alpha)-\{\beta\}$ and $G_{\alpha\gamma\delta}$ fixes $\{\beta\}$, $(\delta)-\{\beta\}$, and $\Gamma(\alpha)-(\delta)$ setwise. In particular $G_{\alpha\gamma\delta}$ is intransitive on $\Gamma(\alpha)-\{\beta\}$. It follows (see [10] Hilfsatz 1) that $z=v-1$ and by an easy character argument $G_{\alpha\gamma\delta}$ has two orbits in $\Gamma(\alpha)-\{\beta\}$, namely $(\delta)-\{\beta\}$ and $\Gamma(\alpha)-(\delta)$ of lengths $k-1$ and $v-k$, respectively. Then $G_{\alpha\beta}$ is an automorphism group of a (possibly degenerate) symmetric balanced incomplete block design $\mathscr{D}'$, the points being the points of $\Gamma(\alpha)-\{\beta\}$, the set of blocks being $\{(\delta)^g-\{\beta\}; g\in G_{\alpha\beta}\}$, and incidence being normal set theoretic inclusion. Two distinct points lie in $\mu=k(k-1)/(v-2)$ common blocks. Since a symmetric design cannot have an automorphism group which is 3-transitive on points (see, for example, [10, Satz 3]), we have:

(3.1) LEMMA. *Under the assumptions of Theorem* 2, *if* G_α *is* 4-*transitive on* $\Gamma(\alpha)$ *then* k *is* 1 *or* 2.

So G_α is not 4-transitive on $\Gamma(\alpha)$ (since $k>2$). It is easy to show that $\mathscr{D}'$ extends to a "symmetric 3-design" (see [5]), $\mathscr{D}''$, the set of points being $\Gamma(\alpha)$, the set of blocks being $\{(\delta)^g : g\in G_\alpha\}$, and incidence being normal set theoretic inclusion. Moreover G_α is a group of automorphisms of $\mathscr{D}''$. By [5] one of the following is true.

(i) $\mathscr{D}'$ is a Hadamard design; $v=2k=4\mu+4$,

(ii) $v-1=(\mu+2)(\mu^2+4\mu+2)$, $k-1=\mu^2+3\mu+1$,

(iii) $v-1=111$, $k-1=111$, $\mu=1$,

(iv) $v-1=495$, $k-1=39$, $\mu=3$.

In cases (iii) and (iv) $\mathscr{D}''$ does not have an automorphism group 3-transitive on points, so case (i) or (ii) is true and Theorem 3 is proved.

Remark. The blocks of the design $\mathscr{D}''$ are in 1–1 correspondence with the $\sim$-equivalence classes of blocks of the design $\mathscr{D}$ defined in Section 2.

4. PROOF OF THEOREM 4

Assume that Hypothesis A is satisfied and that $G_\alpha^{\Gamma(\alpha)}\supseteq A_v$ where $v\geqslant 3$. By [4, Proposition 1.3], $G_\alpha^{\Gamma^*(\alpha)}\supseteq A_v$ so that G_α is 2-primitive on both $\Gamma^*(\alpha)$ and $\Gamma(\alpha)$ (even if v is 3 or 4). By Theorem 1, Hypothesis B is true. By Theorem 2, the conclusion of Theorem 4 is true if $v=3$ so assume that $v\geqslant 4$. Then by Theorem 3, and Theorem 2 for the case $v=4$, it follows that $|\Gamma_3(\alpha)|=v(v-1)^2/k$ where k is 1 or 2; that is either Theorem 4(a) is true or $k=2$. So assume that $k=2$. We define the following two sequences of generalized orbitals of G.

The Γ-sequence and Δ-sequence of generalized orbitals. (a) Let $\Gamma_0=\Delta_0=\{(\alpha,\alpha); \alpha\in\Omega\}$, and let $\Gamma_1=\Gamma$, $\Delta_1=\Gamma^*$.

(b) If Γ_i is defined and nonempty define

$$\Gamma_{i+1}=\begin{cases}\Gamma_i\circ\Gamma^*-(\Gamma_i\cup\Gamma_{i-1}), & \text{if } i \text{ is odd},\\ \Gamma_i\circ\Gamma-(\Gamma_i\cup\Gamma_{i-1}), & \text{if } i \text{ is even}.\end{cases}$$

(c) If Δ_i is defined and nonempty define

$$\Delta_{i+1}=\begin{cases}\Delta_i\circ\Gamma-(\Delta_i\cup\Delta_{i-1}), & \text{if } i \text{ is odd},\\ \Delta_i\circ\Gamma^*-(\Delta_i\cup\Delta_{i-1}), & \text{if } i \text{ is even}.\end{cases}$$

Each Δ_i and Γ_i, if defined and nonempty, is clearly a generalized orbital, and whenever defined, $\Delta_{2i}^*=\Delta_{2i}$, $\Gamma_{2i}^*=\Gamma_{2i}$ and $\Delta_{2i+1}^*=\Gamma_{2i+1}$. We shall refer to these sequences as the Γ-sequence and the Δ-sequence. The proof that Γ is O_v uses the fact that G_α is 3-transitive on $\Gamma(\alpha)$. Therefore we shall show first that $G_\alpha^{\Gamma(\alpha)}$ is not A_4.

(4.1) LEMMA. *If $v=4$ then $G_\alpha^{\Gamma(\alpha)}$ is S_4.*

Proof. Assume that $v=4$ and $G_\alpha^{\Gamma(\alpha)}=A_4$. Let $\delta\in\Gamma_3(\alpha)$ and let $\Gamma^*(\delta)\cap\Gamma_2(\alpha)=\{\gamma,\gamma'\}$ and $\Delta_2(\delta)\cap\Gamma_1(\alpha)=\{\beta,\beta'\}$ where (γ,β) and $(\gamma',\beta')\in\Gamma$. Then $G_{\alpha\delta}$ is a subgroup of $G_{\alpha\{\beta,\beta'\}}$ of index 3 so that $|K(\alpha):K(\alpha)_\delta|$ divides 3. It follows, since $K(\alpha)$ is transitive on $\Gamma^*(\beta)-\{\alpha\}$ that $K(\alpha)_\delta=K(\alpha)_\gamma=K(\alpha)_{\gamma'}$. Thus $K(\alpha)$ acts similarly on all its orbits in $\Gamma_2(\alpha)\cup\Gamma_3(\alpha)$, and so $K(\alpha)\cap K^*(\beta)$ fixes $\Gamma_2(\alpha)\cup\Gamma_3(\alpha)$ pointwise. It follows that $K(\alpha)\cap K^*(\beta)=1$ and hence $K(\alpha)\simeq Z_3$ and $G_\alpha\simeq A_4\times Z_3$.

Note that for all $j\geqslant 3$ such that Γ_j is nonempty, if $\delta\in\Gamma_j(\alpha)$ and $\beta\in\Gamma_1(\alpha)$ is such that $\delta\in\Delta_{j-1}(\beta)$, $\Gamma_1(\delta)\cap\Delta_{j-2}(\beta)\subseteq\Gamma_1(\delta)\cap\Gamma_{j-1}(\alpha)$ so that $|\Gamma_1(\delta)\cap\Gamma_{j-1}(\alpha)|\geqslant 2$ if j is even, and similarly if j is odd $|\Delta_1(\delta)\cap\Gamma_{j-1}(\alpha)|\geqslant 2$. It follows from this that $|\Gamma_j(\alpha)|\leqslant|\Gamma_{j-1}(\alpha)|$ for all $j\leqslant 4$. Let $t\geqslant 3$ be the largest integer such that $|\Gamma_t(\alpha)|=18$ and let d be the largest integer such that Γ_d is nonempty. Then $\Omega=\bigcup_{0\leqslant i\leqslant d}\Gamma_i(\alpha)$ (since the $\Gamma\circ\Gamma^*$ graph is connected and the set $\bigcup_{0\leqslant i\leqslant d}\Gamma_i(\alpha)$ is a union of connected components of this graph), and we have the following list of possibilities.

Case	d	$\lvert\Omega\rvert$	$\lvert\Gamma_{t+1}(\alpha)\rvert$	$\lvert\Gamma_{t+2}(\alpha)\rvert$	$\lvert\Gamma_{t+3}(\alpha)\rvert$
1	t	$18t-19$	—	—	—
2	$t+1$	$18t-7$	12	—	—
3	$t+1$	$18t-10$	9	—	—
4	$t+2$	$18t-3$	12	4	—
5	$t+2$	$18t-4$	12	3	—
6	$t+3$	$18t-2$	12	4	1

As $K(\alpha)$ is the only normal subgroup of G_α of order 3, clearly $K(\alpha) = K^*(\alpha)$. Also if $\gamma \in \Gamma_2(\alpha)$ and $\Gamma(\alpha) \cap \Gamma^*(\gamma) = \{\beta\}$ then $G_{\alpha\gamma} = K(\beta)$. Thus the five subgroups of G_α of order 3 are conjugate in G and form two conjugacy classes of subgroups of G_α of sizes 1 and 4. By [18, 3.6′] $G_\alpha = N_G(K(\alpha))$ has exactly two orbits in $\mathrm{fix}_\Omega K(\alpha)$. It follows that $\mathrm{fix}_\Omega K(\alpha) = \{\alpha\} \cup \Gamma_1(\alpha)$ and hence that Γ is self-paired and $\mathrm{fix}_\Omega K(\beta) = \{\beta\} \cup \Gamma(\beta)$. Thus cases 2, 4, 5, and 6 are impossible as one of $K(\alpha)$, $K(\beta)$ must fix a point of $\Gamma_{t+1}(\alpha)$. Consider case 1. Let $\delta \in \Gamma_3(\alpha)$. Then $G_{\alpha\delta} \simeq Z_2$ and by [18, 3.5], $N_G(G_{\alpha\delta})$ is transitive on $\mathrm{fix}_\Omega G_{\alpha\delta}$. Now $G_{\alpha\delta}$ fixes six points in $\Gamma_i(\alpha)$ for each $i = 3, \ldots, t$ and so $|\mathrm{fix}_\Omega G_{\alpha\delta}| = 6t - 11$ divides $|G| = (18t - 19)36$. It follows that $t = 3$ and $|\Omega| = 35$. A minimal normal subgroup of G must be a nonabelian simple group of order $4 \cdot 5 \cdot 7 \cdot 3^\varepsilon$ where ε is 0, 1, or 2, and no such group exists. Finally consider case 3. Let V be the unique sylow 2-subgroup of G_α. By [18, 3.5] $G_\alpha = N_G(V)$ is transitive on $\mathrm{fix}_\Omega V$ which is impossible as $\mathrm{fix}_\Omega V = \{\alpha\} \cup \Gamma_d(\alpha)$. Thus the lemma is proved.

Now if $v \geqslant 6$ then by Knapp [12, Theorem 6.1], $A_v \times A_{v-1} \leqslant G_\alpha \leqslant S_v \times S_{v-1}$. If v is 4 or 5 then we can show as in the case of A_4 above that $K(\alpha) \cap K^*(\beta) = 1$, where $\beta \in \Gamma(\alpha)$, and then $A_v \times A_{v-1} \leqslant G_\alpha \leqslant S_v \times S_{v-1}$ follows from [2]. Let X, Y be disjoint sets of size v, $v-1$, respectively and let $S_v \times S_{v-1} = \{(g, h); g \in \mathrm{Sym}\, X, h \in \mathrm{Sym}\, Y\}$. We shall identify all points in $\bigcup_{i \leqslant 0} \Gamma_i(\alpha)$ and all points in $\bigcup_{i \leqslant 0} \Delta_i(\alpha)$ with certain subsets of $X \cup Y$ of size $v - 1$ such that G_α acts in the natural way, namely, if a point β in $\bigcup_{i \geqslant 0} \Gamma_i(\alpha)$ or $\bigcup_{i \geqslant 0} \Delta_i(\alpha)$ corresponds to $A \cup B$, where $A \subseteq X$, $B \subseteq Y$, and if $(g, h) \in G_\alpha \cap (S_v \times S_{v-1})$ then $\beta^{(g,h)}$ corresponds $A^g \cup B^h$. We shall give proofs for the Γ-sequence; proofs for the Δ-sequence are analogous.

(4.2) *Identification for* $\bigcup_{i \leqslant 3} \Gamma_i(\alpha)$ *and* $\bigcup_{i \leqslant 3} \Delta_i(\alpha)$.

(i) $\Gamma_0(\alpha) = \Delta_0(\alpha) = \{\alpha\}$ is identified with Y.

(ii) $\Gamma_1(\alpha)$ (and $\Delta_1(\alpha)$) can be identified with $\{A; A \subset X, |A| = v - 1\}$ so that if $A \in \Gamma_1(\alpha)$ (or $\Delta_1(\alpha)$) and $(g, h) \in G_\alpha$ then $A^{(g,h)} = A^g$.

(iii) $\Gamma_2(\alpha)$ (and $\Delta_2(\alpha)$) can be identified with $\{A \cup B; A \subset X, |A| = 1, B \subset Y, |B| = v - 2\}$ so that if $A \cup B \in \Gamma_2(\alpha)$ (or $\Delta_2(\alpha)$) and if $(g, h) \in G_\alpha$ then $(A \cup B)^{(g,h)} = A^g \cup B^h$. If $\gamma = A \cup B \in \Gamma_2(\alpha)$ then $\Gamma_1(\gamma) \cap \Gamma_1(\alpha) = \{X - A\}$. If $\delta = A \cup B \in \Delta_2(\alpha)$ then $\Delta_1(\delta) \cap \Delta_1(\alpha) = \{X - A\}$.

(iv) $\Gamma_3(\alpha)$ (and $\Delta_3(\alpha)$) can be identified with $\{A \cup B;\ A \subset X, |A| = v - 2,\ B \subset Y,\ |B| = 1\}$ so that if $A \cup B \in \Gamma_3(\alpha)$ (or $\Delta_3(\alpha)$) and $(g, h) \in G_\alpha$ then $(A \cup B)^{(g,h)} = A^g \cup B^h$. If $\gamma = A \cup B \in \Gamma_3(\alpha)$ then $\Delta_1(\gamma) \cap \Gamma_2(\alpha) = \{A' \cup (Y - B); A' \subset X - A, |A'| = 1\}$, and $\Delta_2(\gamma) \cap \Gamma_1(\alpha) = \{X - A'; A' \subset X - A,\ |A'| = 1\}$; thus A and B are determined by $\Delta_1(\gamma) \cap \Gamma_2(\alpha)$ and $\Delta_2(\gamma) \cap \Gamma_1(\alpha)$ and hence by G_α and the identifications of (ii) and (iii). If $\delta = A \cup B \in \Delta_3(\alpha)$ then $\Gamma_1(\delta) \cap \Delta_2(\alpha) = \{A' \cup (Y - B);$

$A' \subset X-A, |A'|=1\}$ and $\Gamma_2(\gamma) \cap \Delta_1(\alpha) = \{X-A'; A' \subset X-A, |A'|=1\}$ so that A and B are determined by G_α and the identifications of (ii) and (iii).

Proof for $\Gamma_i(\alpha)$, $i \leqslant 3$. (i) This identification is clearly possible for if $(g,h) \in G_\alpha$, $Y^{(g,h)} = Y^h = Y$.

(ii) Choose β in $\Gamma_1(\alpha)$ and $A \subset X$, $|A| = v-1$, and identify β with A. We can identify $K(\alpha)$ with $\{(1,h);\ h \in \operatorname{Sym} Y\} \cap G_\alpha$, and for any $(g,h) \in G_\alpha$ we identify $\beta^{(g,h)} = \beta^g$ with $A^{(g,h)} = A^g$.

(iii) Choose γ in $\Gamma_2(\alpha)$ and choose $B \subset Y$, $|B| = v-2$. Then $\Gamma_1(\gamma) \cap \Gamma_1(\alpha)$ consists of a unique point, say $\beta = A$ where $A \subset X$, $|A| = v-1$. Identify γ with $(X-A) \cup B$. Then if $(g,h) \in G_\alpha$ we identify $\gamma^{(g,h)}$ with $(X-A^g) \cup B^h$. Clearly $\Gamma_1(\gamma^{(g,h)}) \cap \Gamma_1(\alpha) = \{A^g\}$.

(iv) Let $\delta \in \Gamma_3(\alpha)$. Then $\Delta_2(\delta) \cap \Gamma_1(\alpha)$ is a set of two points, say $\beta_i = A_i$ where $A_i \subset X$, $|A_i| = v-1$ for $i = 1, 2$, and $A_1 \neq A_2$. Also $\Delta_1(\delta) \cap \Gamma_2(\alpha)$ is a set of two points, say $\gamma_i \in \Gamma^*(\beta_i) - \{\alpha\}$, $i = 1, 2$. By part (iii) $\gamma_i = (X-A_i) \cup B_i$ where B_i is a subset of Y of size $v-2$, for $i = 1, 2$. Now $G_{\alpha\delta}$ is a subgroup of $G_{\alpha\{\beta_1,\beta_2\}}$ of index $v-1$ and fixes $\{\gamma_1, \gamma_2\}$. Also the orbit of $G_{\alpha\{\beta_1,\beta_2\}}$ containing γ_1 is $(\Gamma^*(\beta_1) \cup \Gamma^*(\beta_2)) - \{\alpha\}$ of size $2(v-1)$. It is easily shown that $G_{\alpha\delta} = (G_{\alpha\{\beta_1,\beta_2\}})_{\{\gamma_1,\gamma_2\}}$ and contains $G_{\alpha\beta_1\beta_2\gamma_1}$ as a subgroup of index 2. Now $G_{\alpha\{\beta_1,\beta_2\}} \supset G_{\alpha\beta_1\beta_2} \supset K(\alpha)$ and so $|K(\alpha) : K(\alpha)_{\gamma_1\gamma_2}| = |K(\alpha) : K(\alpha) \cap G_{\alpha\beta_1\beta_2\gamma_1}| = |K(\alpha)\, G_{\alpha\beta_1\beta_2\gamma_1} : G_{\alpha\beta_1\beta_2\gamma_1}| = |G_{\alpha\beta_1\beta_2} : G_{\alpha\beta_1\beta_2\gamma_1}| = v-1$. Since $K(\alpha)_{\gamma_1\gamma_2}$ fixes the subset $B_1 \cap B_2$ of Y it follows that $B_1 = B_2 = B$ say. Thus we can identify δ with $(A_1 \cap A_2) \cup (Y-B)$ and part (iv) follows.

For the remainder of the identification we need an inductive proof which is given in the following proposition and which comprises the essence of the proof of Theorem 4.

(4.3) PROPOSITION. *Let j be a positive integer.*

(a) *$\Gamma_{2j}(\alpha)$ is defined and nonempty if and only if $\Delta_{2j}(\alpha)$ is defined and nonempty; and if so then $j \leqslant v-1$ and $\Gamma_{2j}(\alpha)$ and $\Delta_{2j}(\alpha)$ are orbits of G_α. Further,*

(i) *$\Gamma_{2j}(\alpha)$ (and $\Delta_{2j}(\alpha)$) can be identified with $\{A \cup B;\ A \subset X, |A| = j, B \subset Y, |B| = v-1-j\}$ so that if $A \cup B \in \Gamma_{2j}(\alpha)$ (or $\Delta_{2j}(\alpha)$) and $(g,h) \in G_\alpha$, $(A \cup B)^{(g,h)} = A^g \cup B^h$. Thus $k_{2j} = |\Gamma_{2j}(\alpha)| = |\Delta_{2j}(\alpha)| = \binom{v}{j}\binom{v-1}{j}$ and if $\gamma = A \cup B \in \Gamma_{2j}(\alpha)$, $(\Delta_{2j}(\alpha))$, then $G_{\alpha\gamma}$ is the setwise stabilizer of $A \cup B$ in the action of G_α on $X \cup Y$.*

(ii) *If $\gamma = A \cup B \in \Gamma_{2j}(\alpha)$, $(\delta = A \cup B \in \Delta_{2j}(\alpha))$, then $\Gamma(\gamma) \cap \Gamma_{2j-1}(\alpha)$, $(\Delta_1(\delta) \cap \Delta_{2j-1}(\alpha))$, is $\{(X-A) \cup B';\ B' \subset Y-B, |B'| = j-1\}$, and $\Gamma_2(\gamma) \cap \Gamma_{2j-2}(\alpha)$, $(\Delta_2(\gamma) \cap \Delta_{2j-2}(\alpha))$, is $\{A' \cup B';\ A' \subset A, |A-A'| = 1, B \subset B' \subset Y, |B'-B| = 1\}$. Thus $c_{2j} = |\Gamma(\gamma) \cap \Gamma_{2j-1}(\alpha)| = |\Delta_1(\delta) \cap \Delta_{2j-1}(\alpha)| = j$, and if $j > 1$ then A and B are determined by previous identifications.*

(b) *If* $\Gamma_{2j+1}(\alpha)=\Delta^*_{2j+1}(\alpha)$ *is defined and nonempty then* $j\leqslant v-2$ *and* $\Gamma_{2j+1}(\alpha)$ *and* $\Delta_{2j+1}(\alpha)$ *are orbits of* G_α. *Further,*

(i) $\Gamma_{2j+1}(\alpha)$ *(and* $\Delta_{2j+1}(\alpha)$*) can be identified with* $\{A\cup B;\ A\subset X, |A|=v-j-1,\ B\subset Y,\ |B|=j\}$ *so that if* $A\cup B\in\Gamma_{2j+1}(\alpha)$, *(or* $\Delta_{2j+1}(\alpha)$*), and* $(g,h)\in G_\alpha$ *then* $(A\cup B)^{(g,h)}=A^g\cup B^h$. *Thus* $k_{2j+1}=|\Gamma_{2j+1}(\alpha)|=|\Delta_{2j+1}(\alpha)|=\binom{v}{j+1}\binom{v-1}{j}$, *and if* $\gamma=A\cup B\in\Gamma_{2j+1}(\alpha)$, $(\Delta_{2j+1}(\alpha))$, *then* $G_{\alpha\gamma}$ *is the setwise stabilizer of* $A\cup B$ *in the action of* G_α *on* $X\cup Y$.

(ii) *If* $\gamma=A\cup B\in\Gamma_{2j+1}(\alpha)$, $(\delta=A\cup B\in\Delta_{2j+1}(\alpha))$, *then* $\Delta_1(\gamma)\cap\Gamma_{2j}(\alpha)$, $(\Gamma_1(\delta)\cap\Delta_{2j}(\alpha))$, *is* $\{A'\cup(Y-B);\ A'\subset X-A,\ |A'|=j\}$, *and* $\Delta_2(\gamma)\cap\Gamma_{2j-1}(\alpha)$, $(\Gamma_2(\delta)\cap\Delta_{2j-1}(\alpha))$, *is* $\{A'\cup B';\ A\subset A'\subseteq X,\ |A'-A|=1, B'\subset B,\ |B-B'|=1\}$ *Thus* $c_{2j+1}=|\Delta_1(\gamma)\cap\Gamma_{2j}(\alpha)|=|\Gamma_1(\delta)\cap\Delta_{2j}(\alpha)|=j+1$, *and the sets* A *and* B *are determined by previous identifications.*

(c) *If* $\Gamma_i(\alpha)$, $(\Delta_i(\alpha))$, *is the last nonempty member of the* Γ*-sequence, (*Δ*-sequence), and* $\gamma\in\Gamma_i(\alpha)$, $(\delta\in\Delta_i(\alpha))$, *then* $|\Gamma(\gamma)\cap\Gamma_i(\alpha)|$, $(|\Gamma(\delta)\cap\Delta_i(\alpha)|$, *is* $v-c_i=v-[(i+1)/2]$.

Proof. By (4.2) parts (a) and (b) are true for $j=1$. Suppose that $j\geqslant 1$ and that parts (a) and (b) are true for integers less than or equal to j.

Proof of part (a). Assume that $\Gamma_{2j+2}(\alpha)$ is defined and nonempty. First we show that $\Gamma_{2j+2}(\alpha)$ is an orbit of G_α. Let $\gamma=A\cup B\in\Gamma_{2j+1}(\alpha)$. Then $\alpha\in\Delta_{2j+1}(\gamma)$ and by induction it follows that $G_{\alpha\gamma}$ has two orbits in $\Delta_1(\gamma)$, namely, $\Delta_1(\gamma)\cap\Gamma_{2j}(\alpha)$ and its complement in $\Delta_1(\alpha)$. Since $\Gamma_{2j+2}(\alpha)$ is nonempty, $\Delta_1(\gamma)-\Gamma_{2j}(\alpha)$ is a subset of $\Gamma_{2j+2}(\alpha)$ and is an orbit of $G_{\alpha\gamma}$. It follows that G_α is transitive on $\Gamma_{2j+2}(\alpha)$, and an easy counting argument gives $k_{2j+2}c_{2j+2}=k_{2j+1}(v-c_{2j+1})=\binom{v}{j+1}\binom{v-1}{j+1}(j+1)$, where $k_{2j+2}=|\Gamma_{2j+2}(\alpha)|$ and $c_{2j+2}=|\Gamma(\delta)\cap\Gamma_{2j+1}(\alpha)|$ for $\delta\in\Gamma_{2j+2}(\alpha)$. Thus if one of k_{2j+2}, c_{2j+2} is as given in part (a) then both of them are.

Now we show that $\Delta_{2j+2}(\alpha)$ is nonempty. Let $\delta\in\Gamma_{2j+2}(\alpha)$. Then there is a $(2j+3)$-tuple $(\alpha,\gamma_1,\dots,\gamma_{2j+1},\delta)$ where $\gamma_i\in\Gamma_i(\alpha)$ for $1\leqslant i\leqslant 2j+1$, $(\gamma_{2i},\gamma_{2i\pm1})\in\Gamma$ for $i\leqslant j$, and $(\delta,\gamma_{2j+1})\in\Gamma$. Let $\gamma=\gamma_{2j+1}$. Then $\alpha\in\Delta_{2j+1}(\gamma)$, and as, by induction, $|\Gamma(\alpha)\cap\Delta_{2j}(\gamma)|\asymp j<v$, there is a point $\beta\in\Gamma(\alpha)-\Delta_{2j}(\gamma)$. Thus $\beta\in\Delta_{2j+1}\circ\Gamma(\gamma)$. Suppose that $\beta\in\Delta_{2j+1}(\gamma)$. Then there is a $(2j+2)$-tuple $(\beta,\beta_2,\dots,\beta_{2j+1},\gamma)$ where $(\beta_{2i},\beta_{2i\pm1})\in\Gamma$ for $i\leqslant j$, where $\beta=\beta_1$, and $(\gamma,\beta_{2j+1})\in\Gamma$, and all $2j+2$ points are distinct. It follows that for some $i\leqslant 2j+1$, if $\eta\in\Gamma_i(\alpha)$ then $\Gamma_i(\alpha)\cap\Gamma(\eta)$ and $\Gamma_i(\alpha)\cap\Gamma^*(\eta)$ are both nonempty, whereas we know by induction that either $\Gamma(\eta)$ or $\Gamma^*(\eta)$ is contained in $\Gamma_{i-1}(\alpha)\cup\Gamma_{i+1}(\alpha)$. Thus $\beta\notin\Delta_{2j+1}(\gamma)$ and so $\beta\in\Delta_{2j+2}(\gamma)$.

If we interchange Γ and Γ^* in the above arguments we will have shown that Γ_{2j+2} is defined and nonempty if and only if Δ_{2j+2} is; and if so then both are orbits of G_α. We shall give the remainder of the proof of (a) for Γ_{2j+2}; the proof for Δ_{2j+2} is analogous.

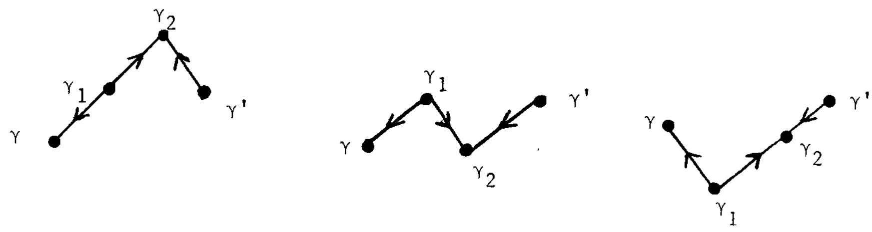

FIGURE 1

In order to identify a point of $\Gamma_{2j+2}(\alpha)$ with the required subset of $X \cup Y$ we investigate $\Gamma_{2j}(\alpha) \cap \Delta_3(\gamma)$ where $\gamma = A \cup B \in \Gamma_{2j+1}(\alpha)$. By induction this set is nonempty so let $\gamma' = A' \cup B' \in \Gamma_{2j}(\alpha) \cap \Delta_3(\gamma)$. (All sets A, B are subsets of X, Y, respectively, with sizes as given in the proposition.) Then there are exactly two 4-tuples $(\gamma, \gamma_1, \gamma_2, \gamma')$ where (γ_1, γ), (γ_1, γ_2), $(\gamma', \gamma_2) \in \Gamma$, $\gamma' \neq \gamma_1$, $\gamma \neq \gamma_2$. Clearly γ_2 lies in $\Gamma_{2j-1}(\alpha)$ or $\Gamma_{2j+1}(\alpha)$ and γ_1 lies in $\Gamma_{2j}(\alpha)$ or $\Gamma_{2j+2}(\alpha)$; there are three possible types of 4-tuples, shown diagrammatically in Fig. 1. (Directed edges are in the Γ-graph.)

Type 1. $\gamma_1 \in \Gamma_{2j}(\alpha)$, $\gamma_2 \in \Gamma_{2j-1}(\alpha)$. Let $\gamma_1 = A_1 \cup B_1$, $\gamma_2 = A_2 \cup B_2$. Then by induction $A_1 = A' = X - A_2 \subset X - A$; since $\gamma_1 \neq \gamma'$, $B_1 \neq B'$ and so $B_1 = Y - B$ and $B_2 = (Y - B') \cap B$. Thus γ_1 and γ_2 are uniquely determined by γ and γ'. Hence there is at most one 4-tuple of type 1 and if there is one then $A \cap A' = \varnothing$ and $|B' \cap B| = 1$.

Type 2. $\gamma_1 \in \Gamma_{2j}(\alpha)$, $\gamma_2 \in \Gamma_{2j+1}(\alpha)$. Let $\gamma_1 = A_1 \cup B_1$, and $\gamma_2 = A_2 \cup B_2$. Then by induction $B = B_2 = Y - B' = Y - B_1$. As $\gamma_1 \neq \gamma'$ and $\gamma_2 \neq \gamma$, $A_1 \neq A'$ and $A_2 \neq A$, and so $A_1 = (X - A) \cap (X - A_2)$ and $X - A_2 \supseteq A'$. It follows from the sizes of the sets that $A \cap A'$ consists of one point, say x, and $X - (A \cup A')$ consists of two points, say y and z. We have two choices for A_2, namely, $(A - \{x\}) \cup \{y\}$ and $(A - \{x\}) \cup \{z\}$, and given A_2, A_1 is uniquely determined. Thus if there is at least one 4-tuple of type 2 then there are exactly two 4-tuples of type 2 and $B' = Y - B$, $|A \cap A'| = 1$.

Type 3. The only other type of 4-tuple has $\gamma_1 \in \Gamma_{2j+2}(\alpha)$, $\gamma_2 \in \Gamma_{2j+1}(\alpha)$. If there is a 4-tuple of type 1 from γ to γ' then as there are exactly two 4-tuples from γ to γ' there must be a 4-tuple of type 3 from γ to γ'. It is also possible that for some γ' there are two 4-tuples of type 3. The number of 4-tuples of type 3 with γ fixed and γ' varying in $\Gamma_{2j}(\alpha) \cap \Delta_3(\gamma)$ is $|\Gamma^*(\gamma) \cap \Gamma_{2j+2}(\alpha)| \cdot (c_{2j+2} - 1)\, c_{2j+1} = (v - j - 1)(c_{2j+2} - 1)(j + 1)$. The number of 4-tuples of type 1 with γ fixed is the number of $\gamma' = A' \cup B' \in \Gamma_{2j}(\alpha)$ with $A \cap A' = \phi$, $|B' \cap B| = 1$, that is, $(j + 1)j(v - j - 1)$. For each 4-tuple of type 1 there is exactly one 4-tuple of type 3. If there are x points γ' for which there are two 4-tuples of type 3 from γ to γ' then we have $(v-j-1)(c_{2j+2}-1)(j+1)=(v-j-1)j(j+1)+2x$, that is, $c_{2j+2}=j+1+2x/(v-j-1)(j+1)\geqslant j+1$.

Now as $\gamma\in\Gamma_{2j+1}(\alpha)$, we have $\alpha\in\Delta_{2j+1}(\gamma)$ and it follows by induction that $G_{\alpha\gamma}$ has exactly four orbits in $\Delta_2(\gamma)$ (for if $\delta\in\Delta_{2j+1}(\alpha)$ is identified with $A\cup B$ then the orbits of $G_{\alpha\delta}$ in $X\cup Y$ are A, $X-A$, B, $Y-B$). The following are three disjoint nonempty subsets of $\Delta_2(\gamma)$ which are fixed setwise by $G_{\alpha\gamma}$: (i) $\Delta_2(\gamma)\cap\Gamma_{2j-1}(\alpha)$, (ii) $\{\eta;\eta\in\Delta_2(\gamma)\cap\Gamma_{2j+1}(\alpha),\ \Delta_1(\gamma)\cap\Delta_1(\eta)\subseteq\Gamma_{2j}(\alpha)\}$, and (iii) $\{\eta;\eta\in\Delta_2(\gamma)\cap\Gamma_{2j+1}(\alpha),\Delta_1(\gamma)\cap\Delta_1(\eta)\subseteq\Gamma_{2j+2}(\alpha)\}$. If Γ_{2j+3} is nonempty then $\Delta_2(\gamma)\cap\Gamma_{2j+3}(\alpha)$ is a fourth nonempty subset of $\Delta_2(\gamma)$ fixed by $G_{\alpha\gamma}$ and so these four sets are orbits of $G_{\alpha\gamma}$. Thus in this case $G_{\alpha\gamma}$ is transitive on the 4-tuples of type 1 and on the 4-tuples of type 3 from γ, and it follows that $c_{2j+2}=j+1$. We can also conclude in this case that $\Gamma_{2j+3}(\alpha)$ is an orbit of G_α. If Γ_{2j+3} is empty and if for $\gamma_1\in\Gamma_{2j+2}(\alpha)$, $\Gamma(\gamma_1)\cap\Gamma_{2j+2}(\alpha)$ is nonempty then $\Delta_2(\gamma)\cap\Gamma_{2j+2}(\alpha)$ is a fourth nonempty subset of $\Delta_2(\gamma)$ fixed by $G_{\alpha\gamma}$. By a similar argument we find $c_{2j+2}=j+1$, and hence that $|\Gamma(\gamma_1)\cap\Gamma_{2j+2}(\alpha)|=v-c_{2j+2}$. Thus if the last nonempty member of the Γ-sequence is Γ_i and i is even then part (c) is true. If Γ_{2j+3} is empty and if for $\gamma_1\in\Gamma_{2j+2}(\alpha)$, $\Gamma(\gamma_1)\cap\Gamma_{2j+2}(\alpha)$ is empty then $\Gamma(\gamma_1)\subseteq\Gamma_{2j+1}(\alpha)$ and it follows that $\{\alpha\}\cup\Gamma_2(\alpha)\cup\Gamma_4(\alpha)\cup\cdots\cup\Gamma_{2j+2}(\alpha)$ is a connected component of the Γ_2-graph, which contradicts [9, (1.12)]. Thus we have shown that $c_{2j+2}=j+1$; if Γ_{2j+2} is the last nonempty member of the Γ-sequence then part (c) is true, and if Γ_{2j+3} is nonempty then G_α is transitive on $\Gamma_{2j+3}(\alpha)$.

If $(\gamma,\gamma_1,\gamma_2,\gamma')$ is a 4-tuple of type 3 then there is also a 4-tuple of type 1 from $\gamma=A\cup B$ to $\gamma'=A'\cup B'$, and $A\cap A'=\varnothing$, $|B\cap B'|=1$. If $\gamma_2=A_2\cup B_2$ then by induction $B'=Y-B_2$ and $A'\cap A_2=\varnothing$. Suppose that $A\neq A_2$. Then we must have $A'=(X-A)\cap(X-A_2)$ so that γ' is uniquely determined by γ and γ_2. However, given γ, γ_1, γ_2 there are $c_{2j+1}=j+1>1$ possibilities for $\gamma'\in\Delta_3(\gamma)\cap\Gamma_{2j}(\alpha)$. Hence $A=A_2$. Now $B\cap B_2=B\cap(Y-B')=B-(B\cap B')=B-\{x\}$ say. The group $G_{\alpha\gamma}$ is the setwise stabilizer of $A\cup B$ in the action of G_α on $X\cup Y$, and it follows that $G_{\alpha\gamma\gamma_2}$ is the setwise stabilizer of the sets $\{x\}$ and $B-B_2=\{y\}$ (say) in the action of $G_{\alpha\gamma}$ on Y. The only proper subgroups of $G_{\alpha\gamma}$ strictly containing $G_{\alpha\gamma\gamma_2}$ are $G_{\alpha\gamma x}$ and $G_{\alpha\gamma y}$. As γ_2 determines a unique γ_1, $G_{\alpha\gamma\gamma_2}\subseteq G_{\alpha\gamma\gamma_1}$, and as $G_{\alpha\gamma}$ is transitive on $\Gamma^*(\gamma)\cap\Gamma_{2j+2}(\alpha)$, $|G_{\alpha\gamma}:G_{\alpha\gamma\gamma_1}|=v-j-1$. It follows that either $G_{\alpha\gamma\gamma_1}=G_{\alpha\gamma y}$, or $j=(v-1)/2$ and $G_{\alpha\gamma\gamma_1}=G_{\alpha\gamma x}$. Suppose that $G_{\alpha\gamma\gamma_1}=G_{\alpha\gamma x}$. Now $A'=X-(A\cup\{a\})$ for some $a\in X-A$. Then if $(\gamma,\gamma_1',\gamma_2',\gamma')$ is the 4-tuple of type 1 from γ to γ', it follows from our investigations of such 4-tuples that $\gamma_1'=(X-(A\cup\{a\}))\cup(Y-B)$ and $\gamma_2'=(A\cup\{a\})\cup(B-\{x\})$. Thus $G_{\alpha\gamma xa}$ fixes γ, γ_1, γ_1', and γ_2'; and $G_{\alpha\gamma xa}$ is transitive on $Y-B$ so that $G_{\alpha\gamma xa}$ does not fix γ_2 or γ' (for $|Y-B|=v-1-j=(v-1)/2>1$ in this case). On the other hand $\gamma_1\in\Delta_3(\gamma_2')$ and the two 4-tuples $(\gamma_2',\gamma_1',\gamma,\gamma_1)$ and

$(\gamma_2', \gamma', \gamma_2, \gamma_1)$ are fixed by $G_{\alpha\gamma x\alpha}$, and this is a contradiction. Thus $G_{\alpha\gamma\gamma_1} = G_{\alpha\gamma y}$. It follows that if we fix γ and γ_1 then all possible $\gamma_2 = A_2 \cup B_2$ in $\Gamma(\gamma_1) \cap \Gamma_{2j+1}(\alpha) - \{\gamma\}$ have $A_2 = A$ and $B_2 - B = \{y\}$ (and there are j choices of $B - B_2$ which give the j points γ_2). It follows that we can identify $\gamma_1 \in \Gamma_{2j+2}(\alpha)$ with $(X - A) \cup (Y - (B \cup \{y\})) = A^* \cup B^*$ say; that the action of G_α is as given in part (a) and that $G_{\alpha\gamma_1}$ is the stabilizer of $A^* \cup B^*$ in $X \cup Y$. Also $j + 1 \leqslant v - 1$ since $|B^*| \geqslant 0$. Moreover $\Gamma(\gamma_1) \cap \Gamma_{2j+1}(\alpha) = \{(X - A^*) \cup B';\ B' \subseteq Y - B^*,\ |B'| = j\}$ and $\Gamma_2(\gamma_1) \cap \Gamma_{2j}(\alpha)$ consists of the points of the form $\gamma' = A' \cup B'$ above, namely, $\{A' \cup B'; A' \subset A^*, |A^* - A'| = 1,\ B^* \subset B',\ |B' - B^*| = 1\}$. Thus (a) is true for Γ_{2j+2} and an analogous proof shows that (a) is also true for Δ_{2j+2}.

Proof of part (b). Assume that Γ_{2j+3} is defined and nonempty; then so is $\Delta_{2j+3} = \Gamma^*_{2j+3}$ and conversely. We showed above that in this case $\Gamma_{2j+3}(\alpha)$ is an orbit of G_α, and an argument analogous to that in part (a) shows that k_{2j+3} is as given in (b) if and only if c_{2j+3} is. We show first that $j + 1 \leqslant v - 2$. By induction $j \leqslant v - 2$, so suppose that $j = v - 2$. Let $\gamma \in \Gamma_{2j+3}(\alpha)$ and let $\beta \in \Gamma_1(\alpha)$ be such that $\gamma \in \Delta_{2j+2}(\beta)$. Then by induction $\Delta_1(\gamma) \cap \Delta_{2j+1}(\beta) \subseteq \Delta_1(\gamma) \cap \Gamma_{2j+2}(\alpha)$ so that $c_{2j+3} = |\Delta_1(\gamma) \cap \Gamma_{2j+2}(\alpha)| \geqslant j + 1 = v - 1$. By part (a) for $j + 1 = v - 1$, $|\Gamma_{2j+2}(\alpha)| = v$ and so $|\Gamma_{2j+3}(\alpha)| = v/c_{2j+3}$, (for $c_{2j+2} = v - 1$). It follows that $c_{2j+3} = v$ and $|\Gamma_{2j+3}(\alpha)| = 1$. Thus G is imprimitive on Ω which is a contradiction. Therefore $j + 1 \leqslant v - 2$. In order to identify points of $\Gamma_{2j+3}(\alpha)$ with subsets of $X \cup Y$ we investigate $\Gamma_{2j+1}(\alpha) \cap \Gamma_3(\gamma)$, where $\gamma = A \cup B \in \Gamma_{2j+2}(\alpha)$. It follows from induction and from our knowledge of Γ_{2j+2} that this set is nonempty, so let $\gamma' = A' \cup B' \in \Gamma_{2j+1}(\alpha) \cap \Gamma_3(\gamma)$. Then there are exactly two 4-tuples $(\gamma, \gamma_1, \gamma_2, \gamma')$ such that (γ, γ_1), (γ_2, γ_1), $(\gamma_2, \gamma') \in \Gamma$, $\gamma \neq \gamma_2$, $\gamma' \neq \gamma_1$. Clearly γ_2 lies in $\Gamma_{2j}(\alpha)$ or $\Gamma_{2j+2}(\alpha)$ and γ_1 lies in $\Gamma_{2j+1}(\alpha)$ or $\Gamma_{2j+3}(\alpha)$, and in general there are three possible types of 4-tuples, shown diagrammatically in Fig. 2.

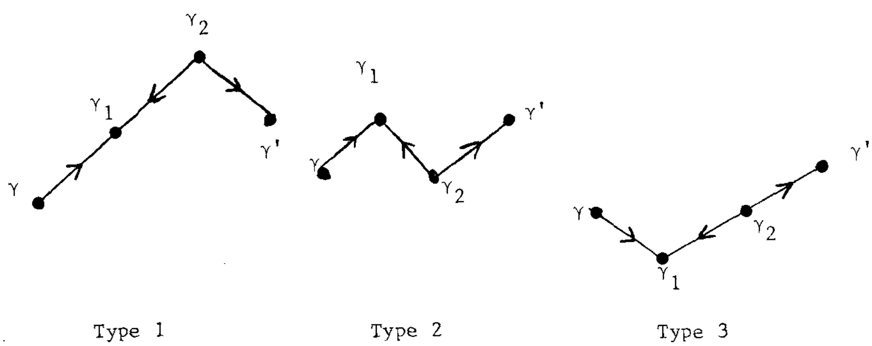

FIGURE 2

Type 1. $\gamma_1 \in \Gamma_{2j+1}(\alpha)$, $\gamma_2 \in \Gamma_{2j}(\alpha)$. Let $\gamma_1 = A_1 \cup B_1$, $\gamma_2 = A_2 \cup B_2$. Then by induction $B_1 = B' = Y - B_2 \subset Y - B$; since $\gamma_1 \neq \gamma'$, $A_1 \neq A'$, and we have $A_1 = X - A$ and $A_2 = (X - A') \cap A$. Thus γ_1 and γ_2 are uniquely determined by γ and γ' and $B' \cap B = \phi$, $|A \cap A'| = 1$.

Type 2. $\gamma_1 \in \Gamma_{2j+1}(\alpha)$, $\gamma_2 \in \Gamma_{2j+2}(\alpha)$. Let $\gamma_1 = A_1 \cup B_1$ and $\gamma_2 = A_2 \cup B_2$. Then by induction $A = A_2 = X - A' = X - A_1$. As $\gamma \neq \gamma_2$, and $\gamma' \neq \gamma_1$, $B \neq B_2$ and $B_1 \neq B'$. Thus $B_1 = (Y - B) \cap (Y - B_2)$ and $B' \subset Y - B_2$. From the sizes of the sets it follows that $B \cap B'$ consists of one point x say, and $Y - (B \cup B')$ consists of two points y and z say. We have two choices for B_2, namely, $(B - \{x\}) \cup \{y\}$ and $(B - \{x\}) \cup \{z\}$, and given B_2, B_1 is determined uniquely. Thus if there is at least one 4-tuple from γ to γ' of type 2, there are two such 4-tuples and $A' = X - A$, $|B \cap B'| = 1$.

Type 3. $\gamma_1 \in \Gamma_{2j+3}(\alpha)$, $\gamma_2 \in \Gamma_{2j+2}(\alpha)$. If there is a 4-tuple of type 1 from γ to γ' then, as there are exactly two 4-tuples from γ to γ', there is a 4-tuple of type 3 from γ to γ'. If γ is fixed and if x is the number of points γ' for which there are two 4-tuples of type 3 from γ to γ', then we can show as in the proof of (a) that

$$c_{2j+3} = j + 2 + 2x/(v - j - 1)(j + 1) \geqslant j + 2.$$

Now $G_{\alpha\gamma}$ is the setwise stabilizer of $A \cup B$ in G_α and so $G_{\alpha\gamma}$ has four orbits in $\Gamma_2(\alpha)$. As Γ_{2j+2} is self-paired, $\alpha \in \Gamma_{2j+2}(\gamma)$ and so $G_{\alpha\gamma}$ has four orbits in $\Gamma_2(\gamma)$. The following are three nonempty disjoint subsets of $\Gamma_2(\gamma)$ fixed setwise by $G_{\alpha\gamma}$: (i) $\Gamma_2(\gamma) \cap \Gamma_{2j}(\alpha)$, (ii) $\{\eta;\ \eta \in \Gamma_2(\gamma) \cap \Gamma_{2j+2}(\alpha),\ \Gamma(\gamma) \cap \Gamma(\eta) \subseteq \Gamma_{2j+1}(\alpha)\}$, and (iii) $\{\eta;\ \eta \in \Gamma_2(\gamma) \cap \Gamma_{2j+2}(\alpha),\ \Gamma(\gamma) \cap \Gamma(\eta) \subseteq \Gamma_{2j+3}(\alpha)\}$. If Γ_{2j+4} is nonempty then $\Gamma_2(\gamma) \cap \Gamma_{2j+4}(\alpha)$ is nonempty and fixed by $G_{\alpha\gamma}$ and so these four sets are orbits of $G_{\alpha\gamma}$. It follows in this case that $G_{\alpha\gamma}$ is transitive on 4-tuples of type 1 and on those of type 3 from γ, and hence that $c_{2j+3} = j + 2$. If Γ_{2j+4} is empty and if for $\gamma_1 \in \Gamma_{2j+3}(\alpha)$, $\Gamma^*(\gamma_1) \cap \Gamma_{2j+3}(\alpha)$ is nonempty, then $\Gamma_2(\gamma) \cap \Gamma_{2j+3}(\alpha)$ is nonempty and fixed by $G_{\alpha\gamma}$. As above $c_{2j+3} = j + 2$, and also $|\Gamma^*(\gamma_1) \cap \Gamma_{2j+3}(\alpha)| = |\Gamma(\gamma_1) \cap \Gamma_{2j+3}(\alpha)| = v - c_{2j+3}$. If Γ_{2j+4} is empty and if for $\gamma_1 \in \Gamma_{2j+3}(\alpha)$, $\Gamma^*(\gamma_1) \cap \Gamma_{2j+3}(\alpha)$ is empty, then $\Gamma^*(\gamma_1) \subseteq \Gamma_{2j+2}(\alpha)$ and it follows that $\{\alpha\} \cup \Gamma_2(\alpha) \cup \Gamma_4(\alpha) \cup \cdots \cup \Gamma_{2j+2}(\alpha)$ is a connected component of the Γ_2-graph, which contradicts [9, (1.12)]. Thus we have shown that $c_{2j+3} = j + 2$, and if Γ_{2j+3} is the last nonempty member of the Γ-sequence then part (c) is true.

If $(\gamma, \gamma_1, \gamma_2, \gamma')$ is a 4-tuple of type 3 then there is also a 4-tuple $(\gamma, \gamma_1', \gamma_2', \gamma')$ of type 1 from γ to γ'. If $\gamma = A \cup B$, $\gamma_2 = A_2 \cup B_2$, $\gamma' = A' \cup B'$, then from the analysis of 4-tuples of type 1 and by induction $A' = X - A_2$, $|A \cap A'| = 1$, and $B' \subseteq (Y - B) \cap (Y - B_2)$. If $B \neq B_2$ then $B' = (Y - B) \cap (Y - B_2)$ so that γ' is uniquely determined by γ and γ_2.

However, given γ, γ_1, γ_2 there are $c_{2j+2} = j+1 > 1$ possibilities for $\gamma' \in \Gamma_3(\gamma) \cap \Gamma_{2j+1}(\alpha)$. Thus $B = B_2$. Now $A \cap A_2 = A \cap (X - A') = A - (A \cap A') = A - \{x\}$ say. The group $G_{\alpha\gamma}$ is the setwise stabilizer of $A \cup B$ in the action of G_α on $X \cup Y$ and it follows that $G_{\alpha\gamma\gamma_2}$ is the setwise stabilizer of the sets $\{x\}$ and $A_2 - A = \{y\}$ (say) in the action of $G_{\alpha\gamma}$ on X. The only proper subgroups of $G_{\alpha\gamma}$ strictly containing $G_{\alpha\gamma\gamma_2}$ are $G_{\alpha\gamma x}$ and $G_{\alpha\gamma y}$. As γ_2 determines a unique γ_1, $G_{\alpha\gamma\gamma_2} \subseteq G_{\alpha\gamma\gamma_1}$, and as $G_{\alpha\gamma}$ is transitive on $\Gamma(\gamma) \cap \Gamma_{2j+3}(\alpha)$, $|G_{\alpha\gamma} : G_{\alpha\gamma\gamma_1}| = v - j - 1$. It follows that either $G_{\alpha\gamma\gamma_1} = G_{\alpha\gamma y}$, or $j = (v-2)/2$ and $G_{\alpha\gamma\gamma_1} = G_{\alpha\gamma x}$. Suppose that $G_{\alpha\gamma\gamma_1} = G_{\alpha\gamma x}$. Now $B' = Y - (B \cup \{b\})$ for some $b \in Y - B$. Thus if $(\gamma, \gamma_1', \gamma_2', \gamma')$ is the 4-tuple of type 1 from γ to γ', we must have $\gamma_1' = (X - A) \cup (Y - (B \cup \{b\}))$, $\gamma_2' = (A - \{x\}) \cup (B \cup \{b\})$ and $\gamma' = (X - (A \cup \{y\}) \cup \{x\}) \cup (Y - (B \cup \{b\}))$. Thus $G_{\alpha\gamma xb}$ fixes γ, γ_1, γ_1', and γ_2', and as it is transitive on $X - A$ it does not fix γ_2 or γ' (for $|X - A| = v - 1 - j = v/2 \geqslant 2$). On the other hand $\gamma_1 \in \Gamma_3(\gamma_2')$ and the two 4-tuples $(\gamma_2', \gamma_1', \gamma, \gamma_1)$ and $(\gamma_2', \gamma', \gamma_2, \gamma_1)$ are fixed by $G_{\alpha\gamma xb}$, which is a contradiction. Thus $G_{\alpha\gamma\gamma_1} = G_{\alpha\gamma y}$. It follows that if we fix γ and γ_1 then all possible $\gamma_2 = A_2 \cup B_2$ in $\Gamma^*(\gamma) \cap \Gamma_{2j+2}(\alpha) - \{\gamma\}$ have $B_2 = B$ and $A_2 - A = \{y\}$ (and there are $j+1$ choices of $A - A_2 = \{x\}$ which give the $j+1$ points γ_2). It follows that we can identify γ_1 with $(X - (A \cup \{y\})) \cup (Y - B) = A^* \cup B^*$ say; that the action of G_α is as given in part (b) and that $G_{\alpha\gamma_1}$ is the stabilizer of $A^* \cup B^*$ in $X \cup Y$. Moreover $\Gamma^*(\gamma_1) \cap \Gamma_{2j+2}(\alpha) = \{A' \cup (Y - B^*);\ A' \subset X - A^*, |A'| = j+1\}$ and $\Gamma_2(\gamma_1) \cap \Gamma_{2j+1}(\alpha) = \{A' \cup B';\ A^* \subset A' \subset X, |A' - A^*| = 1, B' \subset B^*, |B^* - B'| = 1\}$. Thus (b) is true for Γ_{2j+3} and there is analogous proof for Δ_{2j+3}. This completes the proof of Proposition 4.3.

This proposition is now used to obtain the necessary information about the Γ-graph.

(4.4) LEMMA. (a) *The orbital Γ is self-paired.*

(b) *The Γ-graph has diameter $v - 1$.*

(c) *Two points of Ω are joined in the Γ-graph if and only if the subsets of $X \cup Y$ with which they are identified are disjoint.*

Proof. The Γ-sequence is nontrivial since Γ_i is defined and nonempty for $0 \leqslant i \leqslant 3$. Also by Proposition 4.3, Γ_i is empty if $i \geqslant 2v - 1$ and so there is an integer i such that $\Gamma_i(\alpha) \neq \phi$, $\Gamma_{i+1}(\alpha) = \phi$. By Proposition 4.3(c), if $\gamma \in \Gamma_i(\alpha)$ then $|\Gamma(\gamma) \cap \Gamma_i(\alpha)| = v - c_i = v - [(i+1)/2] \geqslant 1$. Now if $v \geqslant 6$ all suborbits of G_α in $\Gamma_i(\alpha)$ are self-paired (for it is easy to show that all suborbits of G_α in its action on the set $\{A \cup B;\ A \subseteq X, |A| = m, B \subseteq Y, |B| = n\}$ are self-paired for any $0 \leqslant m \leqslant v$, $0 \leqslant n \leqslant v - 1$). We can choose $\gamma = A \cup B \in \Gamma_i(\alpha)$ and $\gamma' = A' \cup B' \in \Gamma_i(\alpha) \cap \Gamma(\gamma)$. As all suborbits of G_α in $\Gamma_i(\alpha)$ are self-paired there is an element $g \in G_\alpha$ such that $\gamma^g = \gamma'$ and $\gamma'^g = \gamma$.

Thus $(\gamma', \gamma) = (\gamma^g, \gamma'^g) \in \Gamma$ and hence Γ is self-paired. If v is 4 or 5 then there may be a suborbit of G_α in $\Gamma_i(\alpha)$ which is not self-paired but any such suborbit has length at least $v-1$. The point γ' lies in an orbit of $G_{\alpha\gamma}$ of length at most $v - c_i = v - [(i+1)/2] \leqslant v-2$, so this orbit is self-paired and again we can show that Γ is self-paired.

If $\gamma = A \cup B \in \Gamma_i(\alpha)$ then $G_{\alpha\gamma}$ has two orbits in $\Gamma(\alpha)$ and hence has two orbits in $\Gamma(\gamma)$. It follows that $\Gamma(\gamma) \cap \Gamma_i(\alpha)$ is an orbit of $G_{\alpha\gamma}$ in $\Gamma_i(\alpha)$ of length $v - c_i$, where $0 < v - c_i \leqslant v-2$. Therefore $\Gamma(\gamma) \cap \Gamma_i(\alpha)$ consists of all the points $\gamma' = A' \cup B'$ of $\Gamma_i(\alpha)$ for which $|A \cap A'| = x$, $|B \cap B'| = y$, for some fixed nonnegative integers x and y. Thus if $|A| = m$ and $|B| = n$ then $v - c_i = |\Gamma(\gamma) \cap \Gamma_i(\alpha)| = \binom{m}{x}\binom{v}{m-x}\binom{n}{y}\binom{v-1}{n-y}$. Now it follows from Proposition 4.3 that the Γ-graph has no cycles of *odd* length less than $2i+1$. We claim

(i) If $i = 2j$ then $x = 0$, $B \cup B' = Y$, and $i = v - 1 - y \leqslant v-1$.

(ii) If $i = 2j+1$ then $y = 0$, $A \cup A' = X$, and $i = v-1-x \leqslant v-1$.

Proof of (i). If a point $a \in A \cup A'$ then there are paths of length $i-1$ from γ and γ' to $X - \{a\} \in \Gamma_1(\alpha)$, and hence there is a circuit of length $2i-1$ which is a contradiction. Thus $x = |A \cap A'| = 0$. Suppose that $B \cup B' \subseteq Y - \{b\}$ for some $b \in Y$. As $|A| = j \geqslant 2$ we can choose distinct points a_1, a_2 in A and distinct points a_3, a_4 in A'. Then there are paths of length $i-3$ from γ to $(X - \{a_3, a_4\}) \cup \{b\} \in \Gamma_3(\alpha)$ and from γ' to $(X - \{a_1, a_2\}) \cup \{b\} \in \Gamma_3(\alpha)$. Also $(X - \{a_3, a_4\}) \cup \{b\}$ is joined to $\{a_3\} \cup (Y - \{b\}) \in \Gamma_2(\alpha)$ which is joined to $(X - \{a_1, a_3\}) \cup \{b\} \in \Gamma_3(\alpha)$ which is joined to $\{a_1\} \cup (Y - \{b\}) \in \Gamma_2(\alpha)$ which is joined to $(X - \{a_1, a_2\}) \cup \{b\}$. Thus there is a circuit of length $2i-1$ which is a contradiction. Hence $B \cup B' = Y$ and it follows that $i = 2j = 2|B| = v - 1 - |B \cap B'| = v - 1 - y \leqslant v-1$.

Proof of (ii). By examining the points of $\Gamma_{i-1}(\alpha)$ to which γ and γ' are joined we conclude from the proof of (i) that $(X - A) \cap (X - A') = \phi$, that is $X = A \cup A'$, and hence $i = 2j+1 = v - 1 - |A \cap A'| = v - 1 - x \leqslant v-1$. Also we conclude that $(Y - B) \cup (Y - B') = Y$, that is, $B \cap B' = \phi$.

Thus to complete the proof of the lemma we only need to show that $i = v-1$. Assume that $i \leqslant v-2$, and suppose first that $i = 2j$ is even. Then $m = |A| = j \geqslant 2$, $n = |B| = v-1-j$, and $x = 0$. Since $i \leqslant v-2$, $2 \leqslant j \leqslant v-j-2$ and so $v - 2 \geqslant v - c_j \geqslant \binom{v-m}{m-x} = \binom{v-j}{j} \geqslant \binom{v-j}{2} \geqslant (v+2)v/8$. Thus $v^2 - 6v + 16 \leqslant 0$ which is not true for any $v \geqslant 4$. Thus if $i \leqslant v-2$ then $i = 2j+1$ is odd. Then $m = |A| = v-1-j$, $n = |B| = j$, and $y = 0$. Since $i \geqslant v-2$, $j \geqslant v-j-3$. If $j \geqslant 2$ then $v - 2 \geqslant v - c_i \geqslant \binom{v-1-n}{n-y} = \binom{v-1-j}{j} \geqslant \binom{v-1-j}{2}(v+1)(v-1)/8$ so that v is 4 or 5; however, in this case $v - 2 \geqslant i = 2j+1 \geqslant 5$. Thus $j = 1$, $x = v-1-i = v-4$ and so $v - 2 = v - c_3 = \binom{v-2}{x}\binom{2}{v-2-x}(v-2) = \binom{v-2}{2}(v-2)$. This is impossible for all $v \geqslant i+2 = 5$. Thus $i = v-1$ and Lemma 4.4 is proved.

We have shown that all the vertices of the Γ-graph can be identified with subsets of size $v-1$ of the set $X \cup Y$ of size $2v-1$ (since the Γ-graph is connected), and it follows from Proposition 4.3 and the fact that the diameter is $v-1$ that this correspondence is one-to-one. Further, two vertices are joined if and only if the subsets with which they are identified are disjoint. Thus Γ is the odd graph O_v and G is a subgroup of the group of automorphisms S_{2v-1} of O_v. As $G_\alpha \supseteq A_v \times A_{v-1}$, G is S_{2v-1} or A_{2v-1}. This completes the proof of Theorem 4.

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